

Example 7.28 (Rational function with a **repeated pole**). Find the inverse Laplace transform x of

$$X(s) = \frac{2s+1}{(s+1)^2(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \quad \text{Strictly proper with 1st order pole at } -2 \text{ and 2nd order pole at } -1$$

Solution. To begin, we find a partial fraction expansion of X . We know that X has an expansion of the form

$$X(s) = \underbrace{\frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2}}_{\text{terms from pole at } -1} + \underbrace{\frac{A_{21}}{s+2}}_{\text{term from pole at } -2}$$

Calculating the coefficients of the expansion, we obtain

formula for repeated pole case

coefficient index

pole order

$$A_{11} = \frac{1}{(2-1)!} \left[\left(\frac{d}{ds} \right)^{2-1} [(s+1)^2 X(s)] \right] \Big|_{s=-1} = \frac{1}{1!} \left[\frac{d}{ds} [(s+1)^2 X(s)] \right] \Big|_{s=-1} = \left[\frac{d}{ds} \left(\frac{2s+1}{s+2} \right) \right] \Big|_{s=-1}$$

$$= \left[\frac{(s+2)(2) - (2s+1)(1)}{(s+2)^2} \right] \Big|_{s=-1} = \left[\frac{2s+4-2s-1}{(s+2)^2} \right] \Big|_{s=-1} = \left[\frac{3}{(s+2)^2} \right] \Big|_{s=-1} = 3,$$

$$A_{12} = \frac{1}{(2-2)!} \left[\left(\frac{d}{ds} \right)^{2-2} [(s+1)^2 X(s)] \right] \Big|_{s=-1} = \frac{1}{0!} [(s+1)^2 X(s)] \Big|_{s=-1} = \frac{2s+1}{s+2} \Big|_{s=-1} = \frac{-1}{1} = -1, \quad \text{and}$$

formula for simple pole case

$$A_{21} = (s+2)X(s) \Big|_{s=-2} = \frac{2s+1}{(s+1)^2} \Big|_{s=-2} = \frac{-3}{1} = -3.$$

Thus, X has the expansion

$$\overset{\text{Re}(s) > -1}{\downarrow} X(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2} - \frac{3}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = 3\mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} (t) - \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2} \right\} (t) - 3\mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} (t). \quad (7.7)$$

$\uparrow \text{Re}(s) > -1$ $\uparrow \text{Re}(s) > -1$ $\uparrow \text{Re}(s) > -2$

At this point, it is important to remember that every Laplace transform has an associated ROC, which is an essential component of the Laplace transform. So, when computing the inverse Laplace transform of a function, we must be careful to use the correct ROC for the function. Thus, in order to compute the three inverse Laplace transforms appearing in (7.7), we must associate a ROC with each of the three expressions $\frac{1}{s+1}$, $\frac{1}{(s+1)^2}$, and $\frac{1}{s+2}$. Some care must be exercised in doing so, since each of these expressions has more than one possible ROC and only one is correct. The possible ROCs for each of these expressions is shown in Figure 7.16. In the case of each of these expressions, the correct ROC to use is the one that contains the ROC of X (i.e., $\operatorname{Re}(s) > -1$).

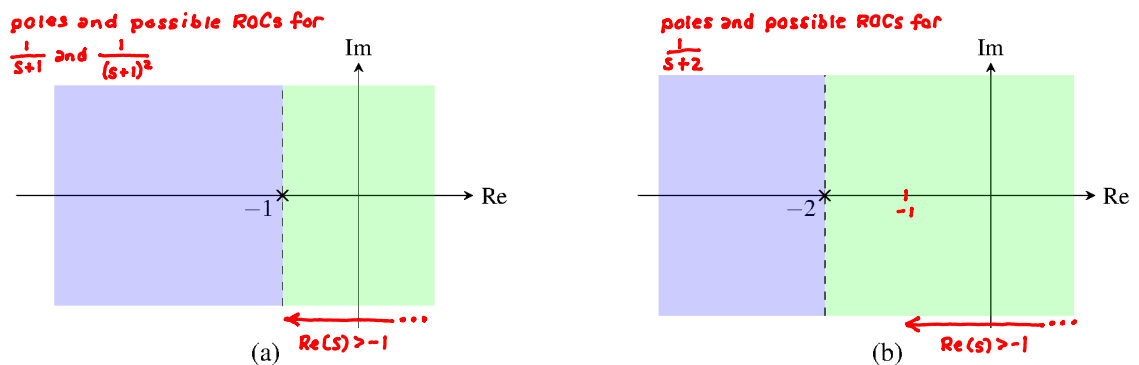


Figure 7.16: The poles and possible ROCs for the rational expressions (a) $\frac{1}{s+1}$ and $\frac{1}{(s+1)^2}$; and (b) $\frac{1}{s+2}$.

LT table



From Table 7.2, we have

$$x(t) = 3L^{-1}\left\{\frac{1}{s+1}\right\}(t) - L^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - 3L^{-1}\left\{\frac{1}{s+2}\right\}(t) \quad (7.7)$$

$\uparrow \text{Re}(s) > -1$ $\uparrow \text{Re}(s) > -1$ $\uparrow \text{Re}(s) > -2$

- ① $e^{-t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+1}$ for $\text{Re}(s) > -1$,
- ② $te^{-t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{(s+1)^2}$ for $\text{Re}(s) > -1$, and
- ③ $e^{-2t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+2}$ for $\text{Re}(s) > -2$.

Substituting these results into (7.7), we obtain

$$\begin{aligned} x(t) &= 3e^{-t}u(t) - te^{-t}u(t) - 3e^{-2t}u(t) \\ &= (3e^{-t} - te^{-t} - 3e^{-2t})u(t). \end{aligned}$$

Substituting ①, ②, and ③ into (7.7)

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