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Lecture Slides for Continuous-Time Signals and Systems (Version: 2013-09-11)

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Part 1

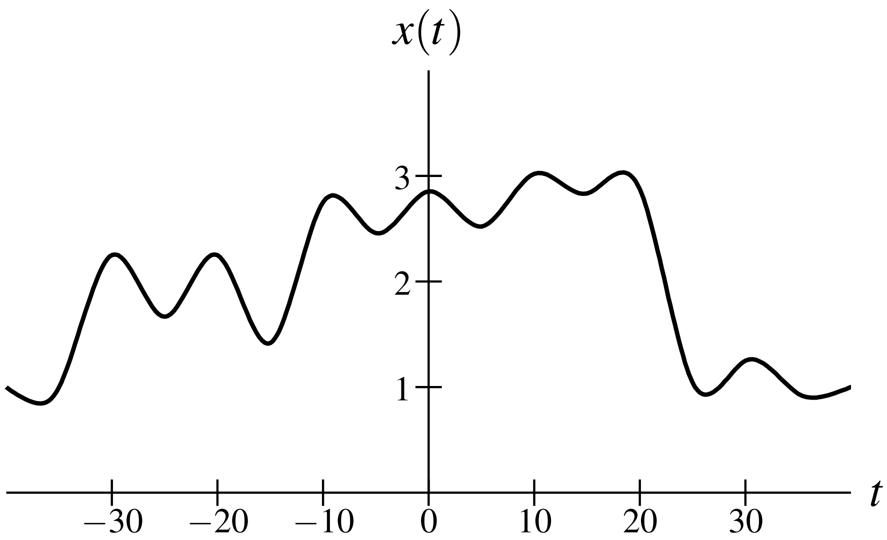
Introduction

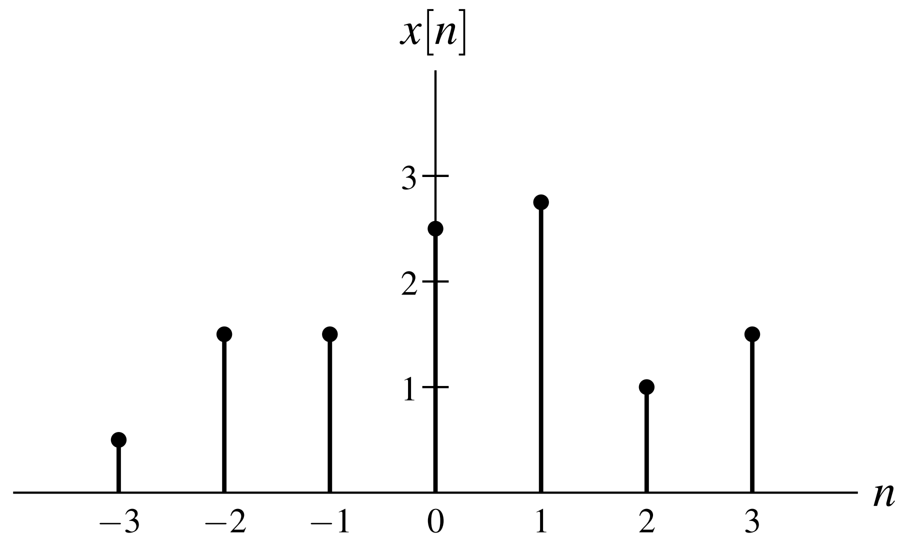
- signal: function of one or more variables that conveys information about some (usually physical) phenomenon
- for function $f(t_1, t_2, \ldots, t_n)$, each of $\{t_k\}$ is called independent variable, function value itself referred to as dependent variable
- examples of signals:
  - voltage or current in electronic circuit
  - position, velocity, and acceleration of object
  - forces or torques in mechanical system
  - flow rates of liquids or gases in chemical process
  - digital image, digital video, digital audio
  - stock market index

Classification of Signals

- number of independent variables (i.e., dimensionality)
  - one: one-dimensional (e.g., audio)
  - more than one: multi-dimensional (e.g., image)
- continuous or discrete independent variables
  - continuous: continuous-time (e.g., voltage waveform)
  - discrete: discrete-time (e.g., stock market index)
- continuous or discrete dependent variable
  - continuous: continuous-valued (e.g., voltage waveform)
  - discrete: discrete-valued (e.g., digital image)
- continuous-valued continuous-time: analog (e.g., voltage waveform)
- discrete-valued discrete-time: digital (e.g., digital audio)

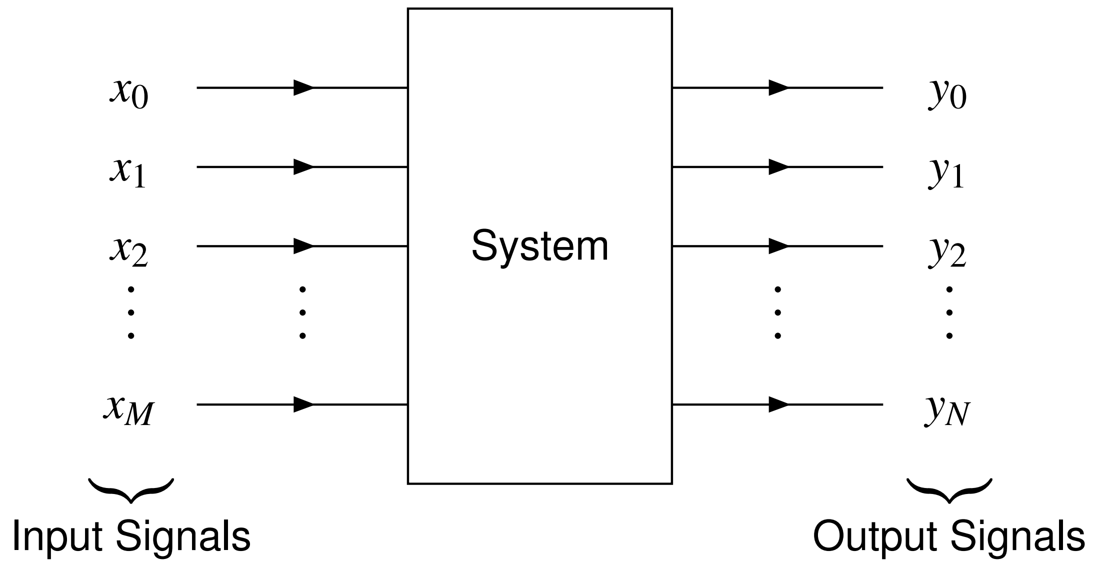
- sequence: discrete-time signal
- independent variables enclosed in parentheses for continuous-time signal (e.g., $x(t)$) and brackets for discrete-time signal (e.g., $x[n]$)





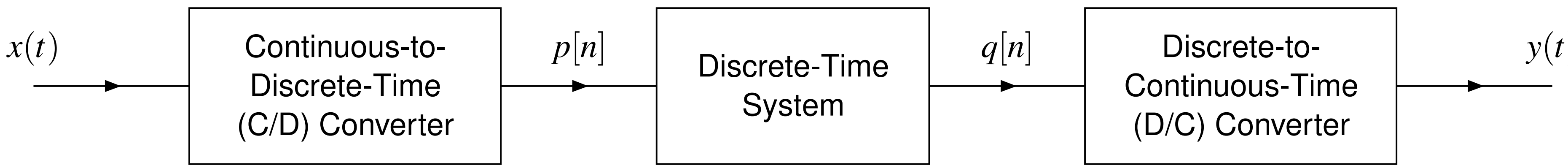
Systems

- system: entity that processes one or more input signals in order to produce one or more output signals

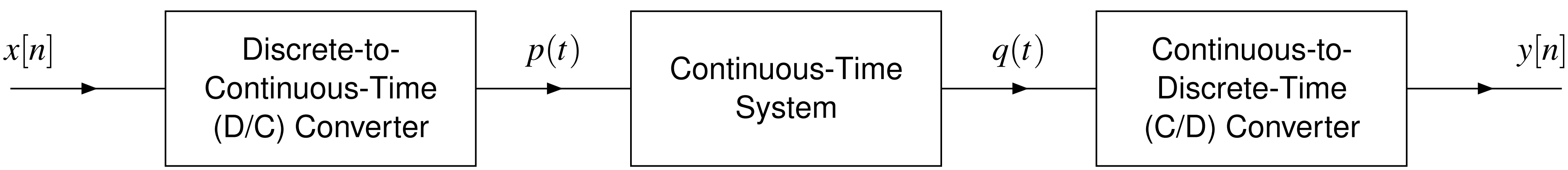


Classification of Systems

- number of inputs
  - single input (SI)
  - multiple input (MI)
- number of outputs
  - single output (SO)
  - multiple output (MO)
- types of signals handled
  - one-dimensional, multi-dimensional
  - continuous-time, discrete-time, hybrid
  - analog, digital



Processing a Continuous-Time Signal With a Discrete-Time System



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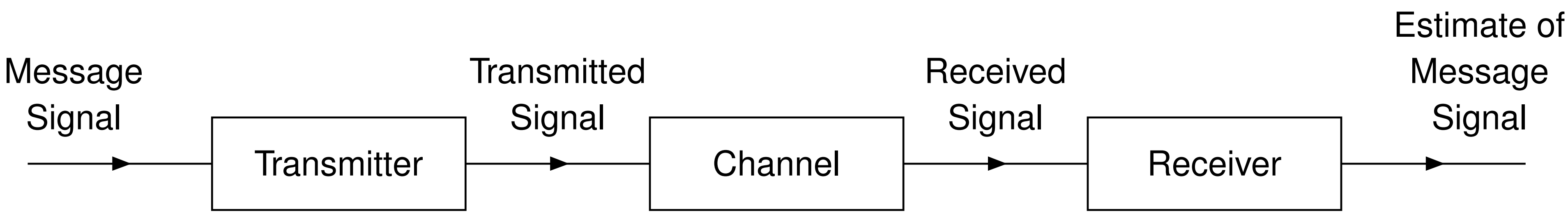
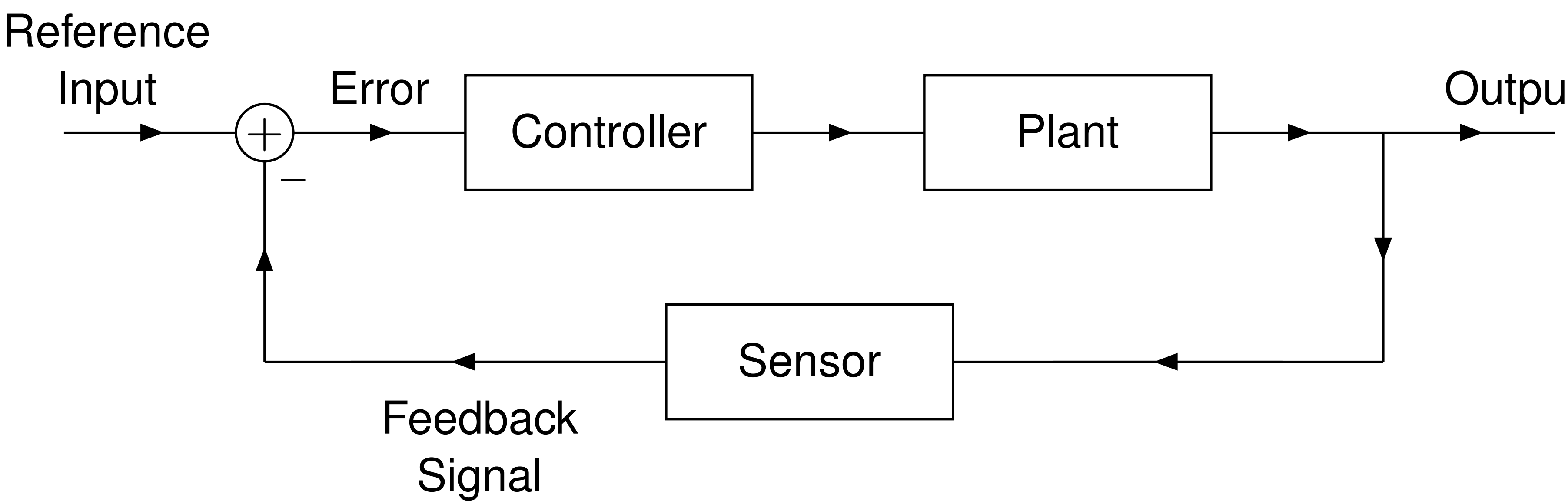


General Structure of a Communication System



General Structure of a Feedback Control System

Why Study Signals and Systems?

- engineers build systems that process/manipulate signals
- need formal mathematical framework for study of such systems
- can use framework to ensure system meets required specifications (performance, safety)
- in this course, focus almost exclusively on continuous-time case (except for sampling, the bridge between continuous time and discrete time)

Some Questions This Course Will Answer

- What is aliasing and why does it occur? (Why do wheels of cars on television often turn in the wrong direction?)
- What is the sampling theorem? Why is CD audio sampled at 44.1 kHz?
- What is a filter? How does an equalizer in stereo system work?
- What is amplitude modulation (AM)? How does AM radio work?
- How can we determine whether a system is stable? If a system is not stable, how can the system be modified to achieve stability?

- suspension bridge linking Tacoma and Gig Harbor (WA, USA)
- mile-long bridge with 2,800 ft. main span, third largest suspension bridge at time of opening
- construction began in Nov. 1938
- about 19 months to build at cost of $6,400,000
- opened to traffic on July 1, 1940
- on Nov. 7, 1940 at approximately 11:00, bridge collapsed during moderate (42 mph) wind storm
- supposed to withstand winds of up to 120 mph
- collapse due to wind-induced vibrations, unstable mechanical system
- repair of bridge was not possible
- fortunately, dog trapped in abandoned car only fatality

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Part 2

Complex Analysis

- A **complex number** is a number of the form $z = x + jy$ where x and y are real numbers and j is the constant defined by $j^2 = -1$ (i.e., $j = \sqrt{-1}$).
- The **Cartesian form** of the complex number z expresses z in the form

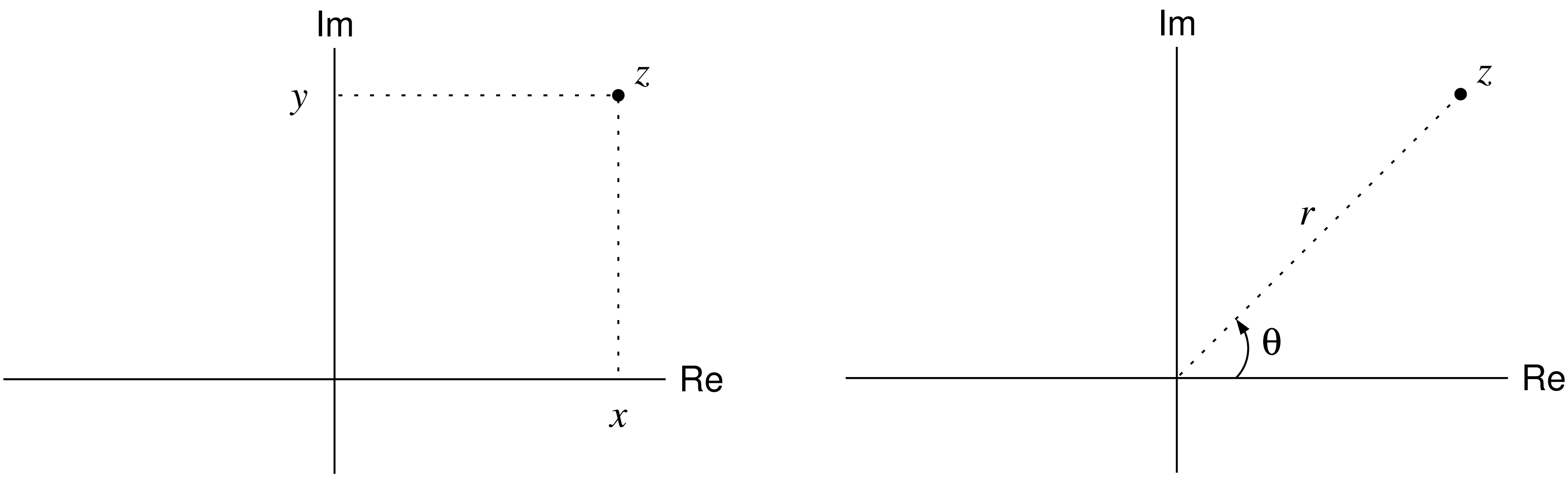
$$z = x + jy,$$

where x and y are real numbers. The quantities x and y are called the **real part** and **imaginary part** of z, and are denoted as $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively.
- The **polar form** of the complex number z expresses z in the form

$$z = r(\cos\theta + j\sin\theta) \quad \text{or equivalently} \quad z = re^{j\theta},$$

where r and θ are real numbers and $r \geq 0$. The quantities r and θ are called the **magnitude** and **argument** of z, and are denoted as $|z|$ and $\arg z$, respectively. [Note: $e^{j\theta} = \cos\theta + j\sin\theta$.]

- Since $e^{j\theta} = e^{j(\theta+2\pi k)}$ for all real θ and all integer k, the argument of a complex number is only uniquely determined to within an additive multiple of 2π.
- The **principal argument** of a complex number z, denoted $\operatorname{Arg} z$, is the particular value θ of $\arg z$ that satisfies $-\pi < \theta \leq \pi$.
- The principal argument of a complex number (excluding zero) is ***unique***.



Cartesian form:

$$z = x + jy$$

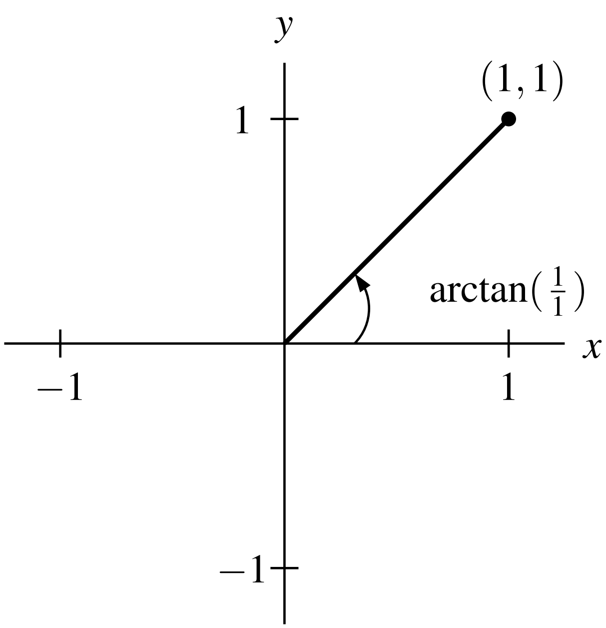
where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$

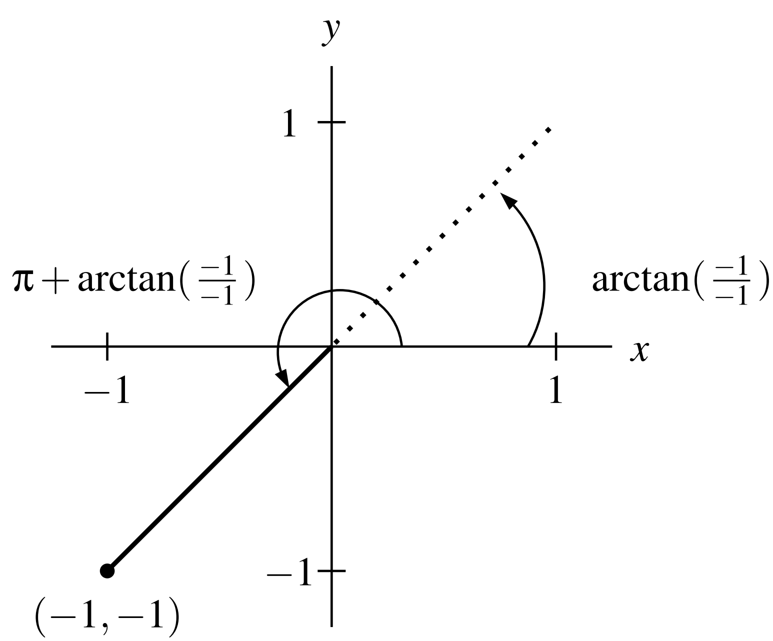
Polar form:

$$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

where $r = |z|$ and $\theta = \arg z$

- The range of the arctan function is $-\pi/2$ (exclusive) to $\pi/2$ (exclusive).
- Consequently, the arctan function always yields an angle in either the first or fourth quadrant.





The atan2 Function

- The angle θ that a vector from the origin to the point (x, y) makes with the positive x axis is given by $\theta = \operatorname{atan2}(y, x)$, where

$$\operatorname{atan2}(y,x) \triangleq \begin{cases} \arctan(y/x) & \text{for } x > 0 \\ \pi/2 & \text{for } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{for } x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & \text{for } x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & \text{for } x < 0 \text{ and } y < 0. \end{cases}$$

- The range of the atan2 function is from $-\pi$ (exclusive) to π (inclusive).
- For the complex number z expressed in Cartesian form $x + jy$, $\operatorname{Arg} z = \operatorname{atan2}(y, x)$.
- Although the atan2 function is quite useful for computing the principal argument (or argument) of a complex number, it is not advisable to memorize the definition of this function. It is better to simply understand what this function is doing (namely, intelligently applying the arctan function).

Conversion Between Cartesian and Polar Form

- Let z be a complex number with the Cartesian and polar form representations given respectively by

$$z = x + jy \quad \text{and} \quad z = re^{j\theta}.$$

- To convert from ***polar to Cartesian*** form, we use the following identities:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

- To convert from ***Cartesian to polar*** form, we use the following identities:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \operatorname{atan2}(y, x) + 2\pi k,$$

  where k is an arbitrary integer.
- Since the $\operatorname{atan2}$ function simply amounts to the intelligent application of the $\arctan$ function, instead of memorizing the definition of the $\operatorname{atan2}$ function, one should simply ***understand*** how to use the $\arctan$ function to achieve the same result.

Properties of Complex Numbers

- For complex numbers, addition and multiplication are ***commutative***. That is, for any two complex numbers z_1 and z_2,

$$z_1 + z_2 = z_2 + z_1 \quad \text{and}$$
$$z_1 z_2 = z_2 z_1.$$

- For complex numbers, addition and multiplication are ***associative***. That is, for any two complex numbers z_1 and z_2,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \text{and}$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

- For complex numbers, the ***distributive*** property holds. That is, for any three complex numbers z_1, z_2, and z_3,

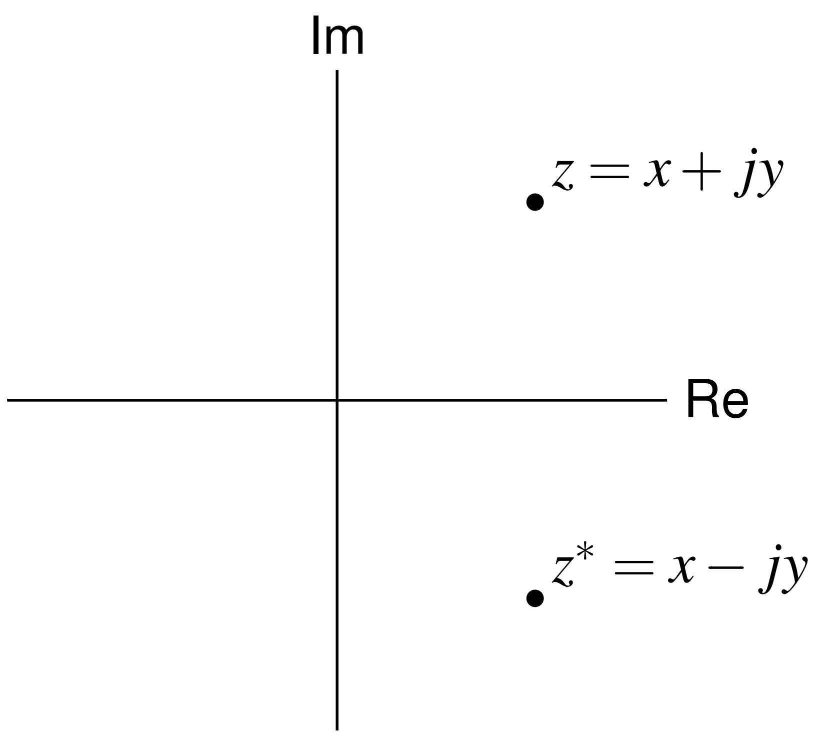
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Conjugation

- The **conjugate** of the complex number $z = x + jy$ is denoted as z^* and defined as

$$z^* = x - jy.$$

- Geometrically, the conjugation operation reflects a point in the complex plane about the real axis.
- The geometric interpretation of the conjugate is illustrated below.



Properties of Conjugation

- For every complex number z, the following identities hold:

$$
\begin{aligned}
\left|z^*\right| &= |z|, \\
\arg z^* &= -\arg z, \\
zz^* &= |z|^2, \\
\operatorname{Re} z &= \tfrac{1}{2}(z+z^*), \quad \text{and} \\
\operatorname{Im} z &= \tfrac{1}{2j}(z-z^*).
\end{aligned}
$$

- For all complex numbers z_1 and z_2, the following identities hold:

$$
\begin{aligned}
(z_1+z_2)^* &= z_1^*+z_2^*, \\
(z_1z_2)^* &= z_1^*z_2^*, \quad \text{and} \\
(z_1/z_2)^* &= z_1^*/z_2^*.
\end{aligned}
$$

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2). \end{aligned}$$

- That is, to add complex numbers expressed in Cartesian form, we simply add their real parts and add their imaginary parts.
- ***Polar form:*** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= (r_1 \cos\theta_1 + jr_1 \sin\theta_1) + (r_2 \cos\theta_2 + jr_2 \sin\theta_2) \\ &= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2). \end{aligned}$$

- That is, to add complex numbers expressed in polar form, we first rewrite them in Cartesian form, and then add their real parts and add their imaginary parts.
- For the purposes of addition, it is easier to work with complex numbers expressed in Cartesian form.

Multiplication

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\begin{aligned} z_1z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1x_2 + jx_1y_2 + jx_2y_1 - y_1y_2 \\ &= (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1). \end{aligned}$$

- That is, to multiply two complex numbers expressed in Cartesian form, we use the distributive law along with the fact that $j^2 = -1$.
- ***Polar form:*** Let $z_1 = r_1e^{j\theta_1}$ and $z_2 = r_2e^{j\theta_2}$. Then,

$$z_1z_2 = \left(r_1e^{j\theta_1}\right)\left(r_2e^{j\theta_2}\right) = r_1r_2e^{j(\theta_1+\theta_2)}.$$

- That is, to multiply two complex numbers expressed in polar form, we use exponent rules.
- For the purposes of multiplication, it is easier to work with complex numbers expressed in polar form.

Division

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2}$$
$$= \frac{x_1 x_2 - jx_1 y_2 + jx_2 y_1 + y_1 y_2}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}.$$

- That is, to compute the quotient of two complex numbers expressed in polar form, we convert the problem into one of division by a real number.
- ***Polar form:*** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}.$$

- That is, to compute the quotient of two complex numbers expressed in polar form, we use exponent rules.
- For the purposes of division, it is easier to work with complex numbers expressed in polar form.

Properties of the Magnitude and Argument

- For any complex numbers z_1 and z_2, the following identities hold:

$$|z_1 z_2| = |z_1|\,|z_2|\,,$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{for } z_2 \neq 0,$$

$$\arg z_1 z_2 = \arg z_1 + \arg z_2, \quad \text{and}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0.$$

- The above properties trivially follow from the polar representation of complex numbers.

Euler's Relation, and De Moivre's Theorem

- **Euler's relation.** For all real θ,

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

- From Euler's relation, we can deduce the following useful identities:

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{and}$$
$$\sin\theta = \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}).$$

- **De Moivre's theorem.** For all real θ and all ***integer*** n,

$$e^{jn\theta} = \left(e^{j\theta}\right)^n.$$

[Note: This relationship does not necessarily hold for ***real*** n.]

Roots of Complex Numbers

- Every complex number $z = re^{j\theta}$ (where $r = |z|$ and $\theta = \arg z$) has n distinct ***nth roots*** given by

$$\sqrt[n]{r}e^{j(\theta+2\pi k)/n} \quad \text{for } k = 0, 1, \ldots, n-1.$$

- For example, 1 has the two distinct square roots 1 and -1.

Quadratic Formula

- Consider the equation

$$az^2 + bz + c = 0,$$

where a, b, and c are real, z is complex, and $a \neq 0$.

- The roots of this equation are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- This formula is often useful in factoring quadratic polynomials.
- The quadratic $az^2 + bz + c$ can be factored as $a(z - z_0)(z - z_1)$, where

$$z_0 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Complex Functions

- A **complex function** maps complex numbers to complex numbers. For example, the function $F(z) = z^2 + 2z + 1$, where z is complex, is a complex function.
- A complex **polynomial function** is a mapping of the form

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

  where z, $a_0, a_1, \ldots, a_n$ are complex.
- A complex **rational function** is a mapping of the form

$$F(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m},$$

  where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ and z are complex.
- Observe that a polynomial function is a special case of a rational function.
- In the context of this course, we will mostly focus our attention on polynomial and rational functions.

- A function $F(z)$ is said to be **continuous at a point** z_0 if $F(z_0)$ is defined and given by

$$F(z_0) = \lim_{z \to z_0} F(z).$$

- A function that is continuous at every point in its domain is said to be **continuous**.
- Polynomial functions are continuous everywhere.
- Rational functions are continuous everywhere except at points where the denominator polynomial becomes zero.

Differentiability

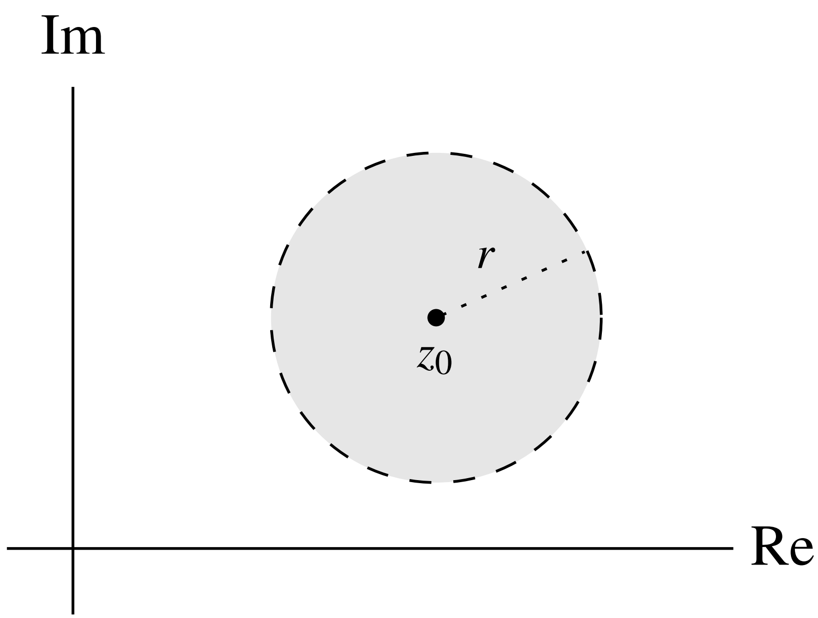
- A function $F(z)$ is said to be **differentiable at a point** $z = z_0$ if the limit
  $$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0}$$
  exists. This limit is called the **derivative** of $F(z)$ at the point $z = z_0$.
- A function is said to be **differentiable** if it is differentiable at every point in its domain.
- The rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. If $F'(z_0)$ and $G'(z_0)$ exist, then
  1. $(aF)'(z_0) = aF'(z_0)$ for any complex constant a;
  2. $(F+G)'(z_0) = F'(z_0) + G'(z_0)$;
  3. $(FG)'(z_0) = F'(z_0)G(z_0) + F(z_0)G'(z_0)$;
  4. $(F/G)'(z_0) = \frac{G(z_0)F'(z_0) - F(z_0)G'(z_0)}{G(z_0)^2}$; and
  5. if $z_0 = G(w_0)$ and $G'(w_0)$ exists, then the derivative of $F(G(z))$ at w_0 is $F'(z_0)G'(w_0)$ (i.e., the chain rule).
- A polynomial function is differentiable everywhere.
- A rational function is differentiable everywhere except at the points where its denominator polynomial becomes zero.

- An **open disk** in the complex plane with center z_0 and radius r is the set of complex numbers z satisfying

$$|z - z_0| < r,$$

where r is a strictly positive real number.

- A plot of an open disk is shown below.



Analyticity

- A function is said to be **analytic at a point** z_0 if it is differentiable at every point in an open disk about z_0.
- A function F is said to be **analytic** if it is analytic at every point in its domain.
- A polynomial function is analytic everywhere.
- A rational function is analytic everywhere, except at the points where its denominator polynomial becomes zero.

- If a function F is zero at the point z_0 (i.e., $F(z_0) = 0$), F is said to have a **zero** at z_0.
- If a function F is such that $F(z_0) = 0, F^{(1)}(z_0) = 0, \ldots, F^{(n-1)}(z_0) = 0$ (where $F^{(k)}$ denotes the kth order derivative of F), F is said to have an ***n*th order zero** at z_0.
- A point at which a function fails to be analytic is called a **singularity**.
- Polynomials do not have singularities.
- Rational functions can have a type of singularity called a pole.
- If a function F is such that $1/F(z)$ has an nth order zero at z_0, F is said to have an ***n*th order pole** at z_0.
- A pole of first order is said to be **simple**, whereas a pole of order two or greater is said to be **repeated**. A similar terminology can also be applied to zeros (i.e., **simple zero** and **repeated zero**).

Zeros and Poles of a Rational Function

- Given a rational function F, we can always express F in factored form as

$$F(z) = \frac{K(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_M)^{\alpha_M}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_N)^{\beta_N}},$$

  where K is complex, $a_1, a_2, \ldots, a_M, b_1, b_2, \ldots, b_N$ are distinct complex numbers, and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $\beta_1, \beta_2, \ldots, \beta_N$ are strictly positive integers.
- One can show that $F(z)$ has ***poles*** at $b_1, b_2, \ldots, b_N$ and ***zeros*** at $a_1, a_2, \ldots, a_M$.
- Furthermore, the kth pole (i.e., b_k) is of ***order*** β_k, and the kth zero (i.e., a_k) is of ***order*** α_k.
- When plotting zeros and poles in the complex plane, the symbols “o” and “x” are used to denote zeros and poles, respectively.

Part 3

Continuous-Time Signals and Systems

Section 3.1

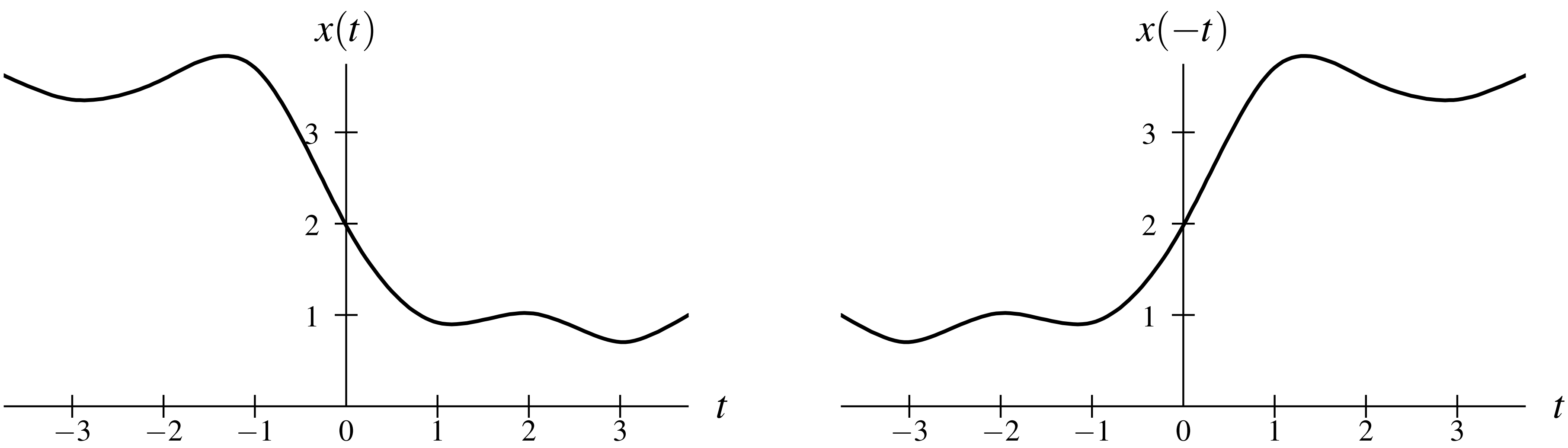
Independent- and Dependent-Variable Transformations

Time Reversal (Reflection)

- **Time reversal** (also known as **reflection**) maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(-t).$$

- Geometrically, the output signal $y(t)$ is a reflection of the input signal $x(t)$ about the (vertical) line $t = 0$.



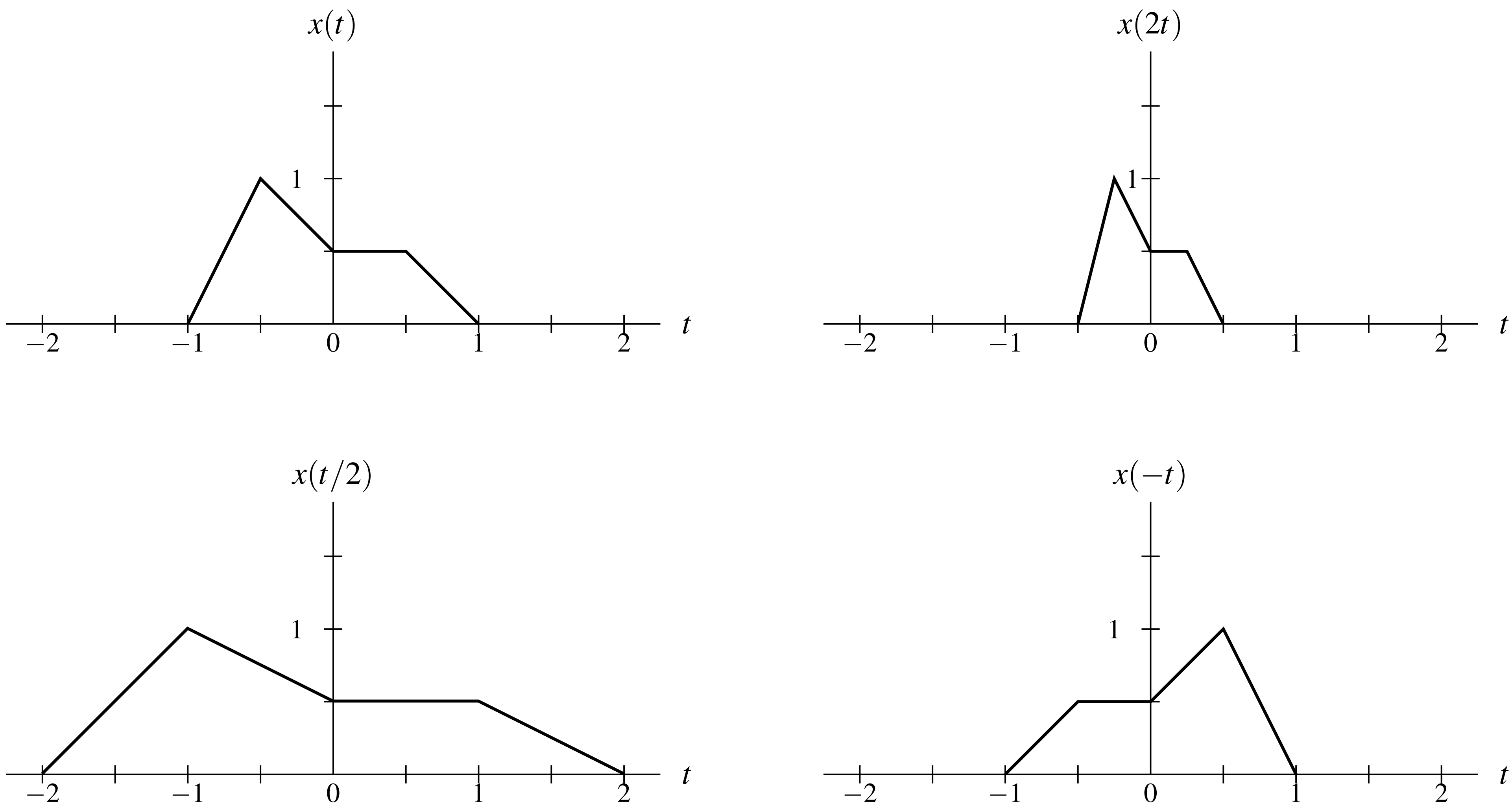
Time Scaling (Dilation/Reflection)

- **Time scaling** (also called **dilation**) maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(at),$$

  where a is a nonzero real number.
- Geometrically, this transformation is associated with a dilation (i.e., compression/expansion along the time axis) and/or reflection about the (vertical) line $t = 0$.
- If $|a| < 1$, the signal is ***expanded*** (i.e., stretched) along the time axis.
- If $|a| > 1$, the signal is instead ***compressed***.
- If $|a| = 1$, the signal is neither expanded nor compressed.
- If $a < 0$, the signal is also ***reflected*** about the vertical line $t = 0$.
- Time reversal is simply a special case of time scaling with $a = -1$.

Time Scaling (Dilation/Reflection): Example



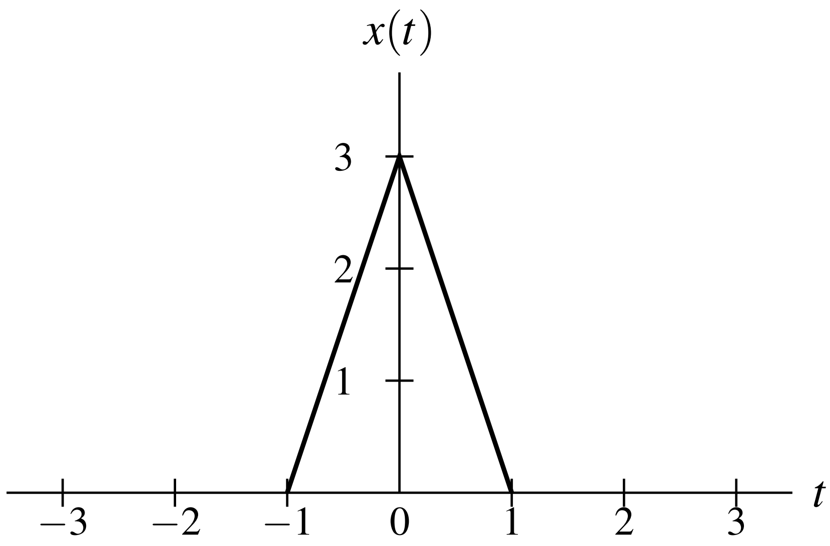
- **Time shifting** (also called **translation**) maps the input signal $x(t)$ to the output signal $y(t)$ as given by

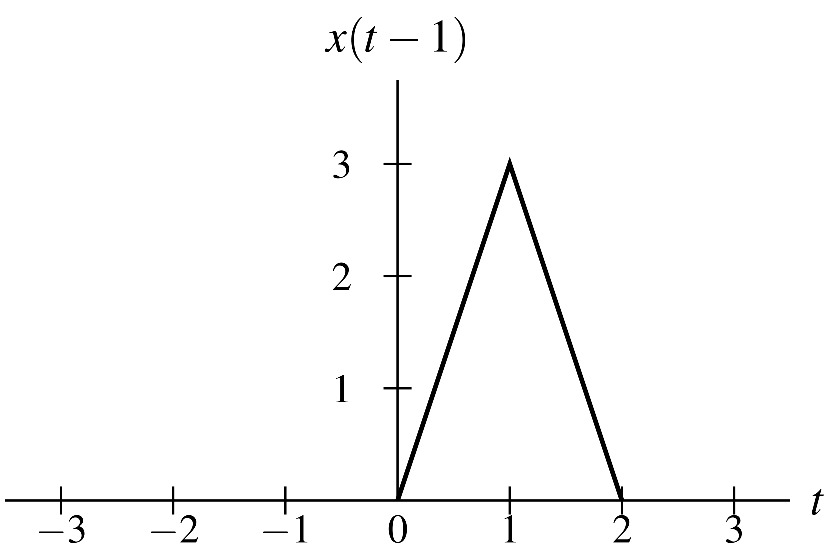
$$y(t) = x(t-b),$$

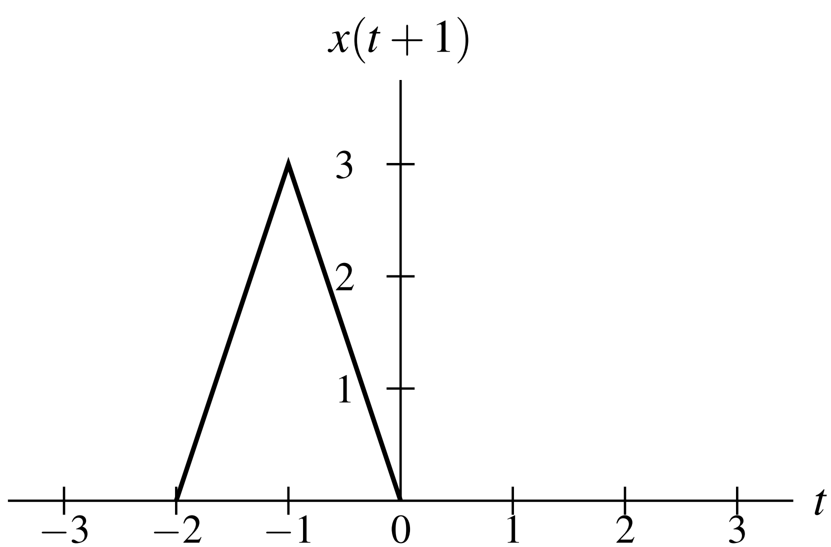
where b is a real number.

- Geometrically, this operation shifts the signal (to the left or right) along the time axis.
- If b is positive, $y(t)$ is ***shifted to the right*** relative to $x(t)$ (i.e., delayed in time).
- If b is negative, $y(t)$ is ***shifted to the left*** relative to $x(t)$ (i.e., advanced in time).

Time Shifting (Translation): Example







Combined Time Scaling and Time Shifting

- Consider a transformation that maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(at - b),$$

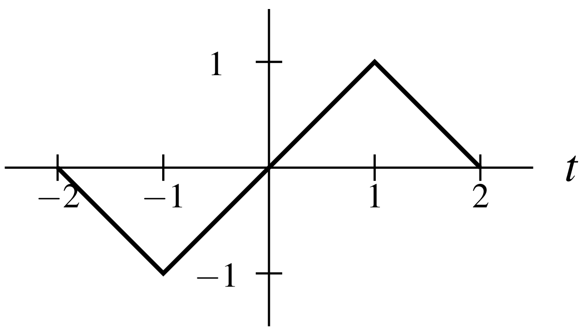
where a and b are real numbers and $a \neq 0$.

- The above transformation can be shown to be the combination of a time-scaling operation and time-shifting operation.
- Since time scaling and time shifting ***do not commute***, we must be particularly careful about the order in which these transformations are applied.
- The above transformation has two distinct but equivalent interpretations:
  1. first, time shifting $x(t)$ by b, and then time scaling the result by a;
  2. first, time scaling $x(t)$ by a, and then time shifting the result by b/a.
- Note that the time shift is not by the same amount in both cases.

Combined Time Scaling and Time Shifting: Example

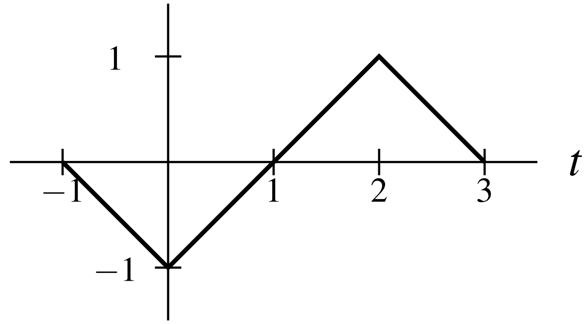
Given $x(t)$ as shown below, find $x(2t-1)$.

$x(t)$

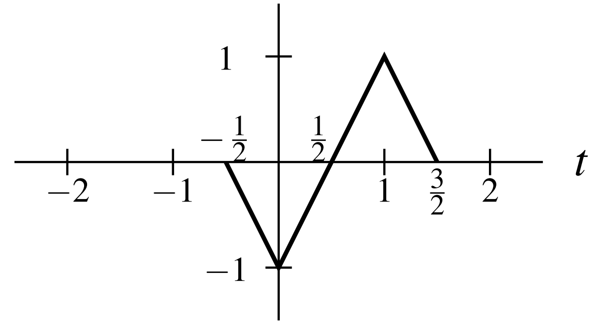


time shift by 1 and then time scale by 2

$p(t) = x(t-1)$



$p(2t) = x(2t-1)$



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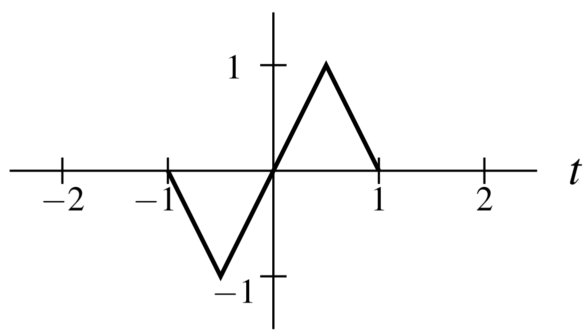
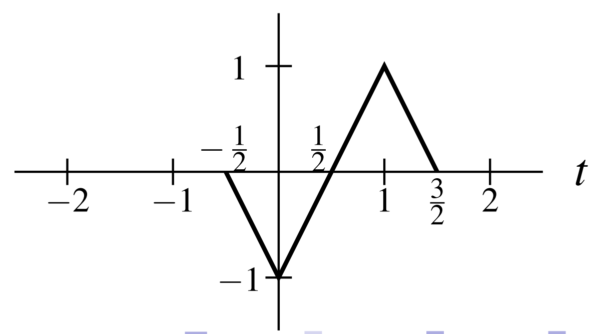
time scale by 2 and then time shift by $\frac{1}{2}$

$q(t) = x(2t)$



$q(t-1/2) = x(2(t-1/2))$
$= x(2t-1)$



Two Perspectives on Independent-Variable Transformations

- A transformation of the independent variable can be viewed in terms of
  1. the effect that the transformation has on the ***signal***; or
  2. the effect that the transformation has on the ***horizontal axis***.
- This distinction is important because such a transformation has ***opposite*** effects on the signal and horizontal axis.
- For example, the (time-shifting) transformation that replaces t by $t-b$ (where b is a real number) in $x(t)$ can be viewed as a transformation that
  1. shifts the signal $x(t)$ ***right*** by b units; or
  2. shifts the horizontal axis ***left*** by b units.
- In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the ***signal***.
- If one is not careful to consider that we are interested in the signal perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

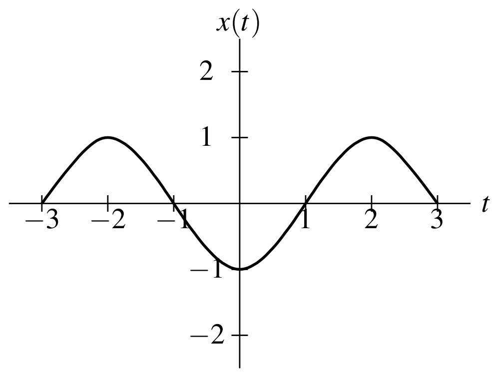
Amplitude Scaling

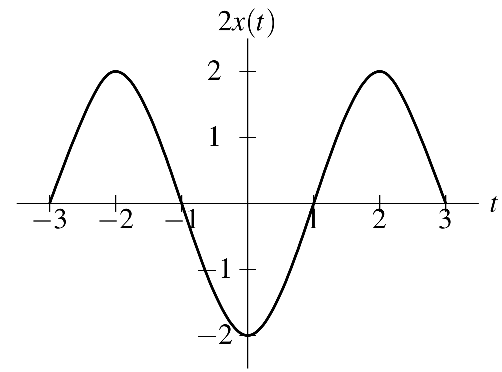
- **Amplitude scaling** maps the input signal $x(t)$ to the output signal $y(t)$ as given by

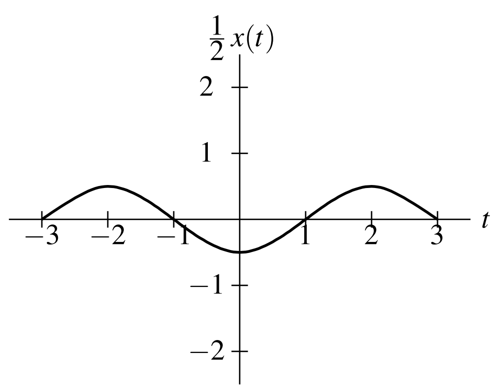
$$y(t) = ax(t),$$

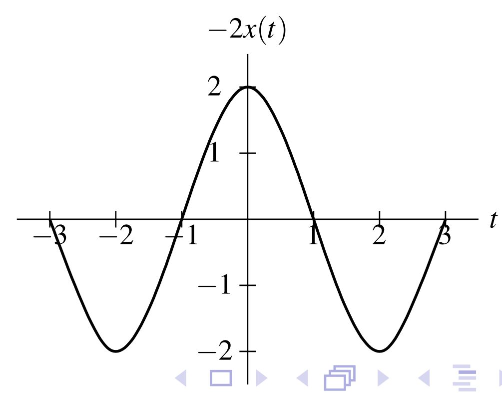
where a is a real number.

- Geometrically, the output signal $y(t)$ is ***expanded/compressed*** in amplitude and/or ***reflected*** about the time axis.









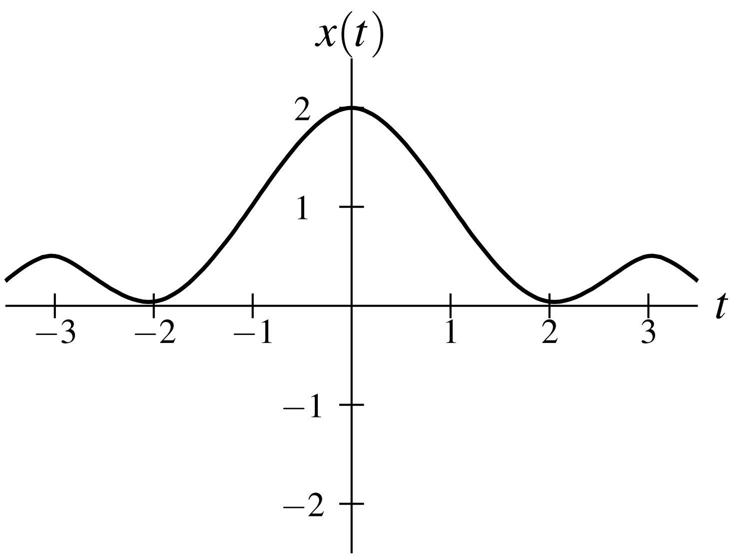
Amplitude Shifting

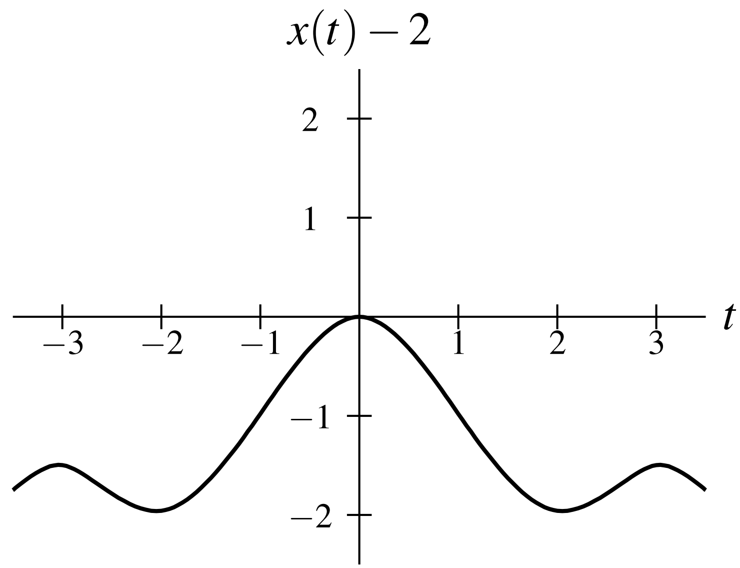
- **Amplitude shifting** maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(t) + b,$$

  where b is a real number.

- Geometrically, amplitude shifting adds a ***vertical displacement*** to $x(t)$.





Combined Amplitude Scaling and Amplitude Shifting

- We can also combine amplitude scaling and amplitude shifting transformations.
- Consider a transformation that maps the input signal $x(t)$ to the output signal $y(t)$, as given by
$$y(t) = ax(t) + b,$$
where a and b are real numbers.
- Equivalently, the above transformation can be expressed as
$$y(t) = a\left[x(t) + \tfrac{b}{a}\right].$$
- The above transformation is equivalent to:
  1. first amplitude scaling $x(t)$ by a, and then amplitude shifting the resulting signal by b; or
  2. first amplitude shifting $x(t)$ by b/a, and then amplitude scaling the resulting signal by a.

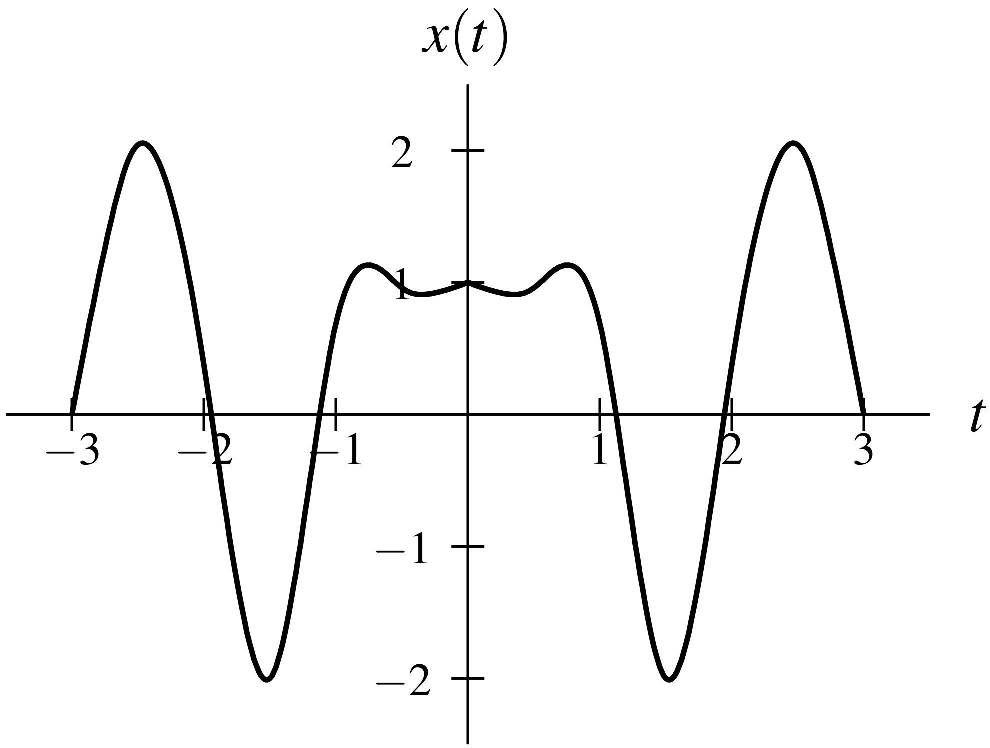
Section 3.2

Properties of Signals

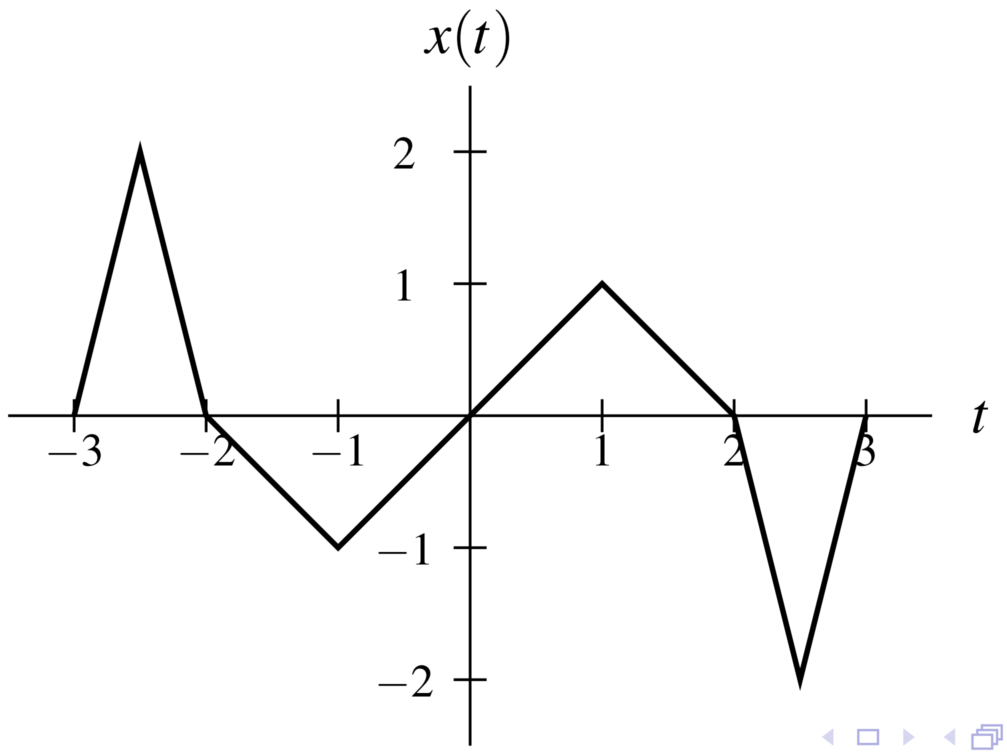
- A signal $x(t)$ is said to be **even** if it satisfies

$$x(t) = x(-t) \quad \text{for all } t.$$

- Geometrically, an even signal is ***symmetric*** about the vertical line $t = 0$.
- Some common examples of even signals include the cosine, absolute value, and square functions.
- Another example of an even signal is shown below.



- A signal $x(t)$ is said to be **odd** if it satisfies
$$x(t) = -x(-t) \quad \text{for all } t.$$
- Geometrically, an odd signal is ***antisymmetric*** about $t = 0$.
- An odd signal $x(t)$ must be such that $x(0) = 0$.
- Some common examples of odd signals include the sine, signum, and cube (i.e., $x(t) = t^3$) functions.
- Another example of an odd signal is shown below.



- Sums involving even and odd signals have the following properties:
  - The sum of two even signals is even.
  - The sum of two odd signals is odd.
  - The sum of an even signal and odd signal is neither even nor odd.
- That is, the ***sum*** of functions with the ***same type of symmetry*** also has the ***same type of symmetry***.
- Products involving even and odd signals have the following properties:
  - The product of two even signals is even.
  - The product of two odd signals is even.
  - The product of an even signal and an odd signal is odd.
- That is, the ***product*** of functions with the ***same type of symmetry*** is ***even***, while the ***product*** of functions with ***opposite types of symmetry*** is ***odd***.

- Any signal $x(t)$ has a ***unique*** representation of the form

$$x(t) = x_e(t) + x_o(t),$$

  where $x_e(t)$ and $x_o(t)$ are ***even*** and ***odd***, respectively.
- In particular, $x_e(t)$ and $x_o(t)$ are given by

$$x_e(t) = \tfrac{1}{2}\left(x(t) + x(-t)\right) \quad \text{and} \quad x_o(t) = \tfrac{1}{2}\left(x(t) - x(-t)\right).$$

- The functions $x_e(t)$ and $x_o(t)$ are called the **even part** and **odd part** of $x(t)$, respectively.
- For convenience, the even and odd parts of $x(t)$ are often denoted as $\mathrm{Even}\{x(t)\}$ and $\mathrm{Odd}\{x(t)\}$, respectively.

Periodic Signals

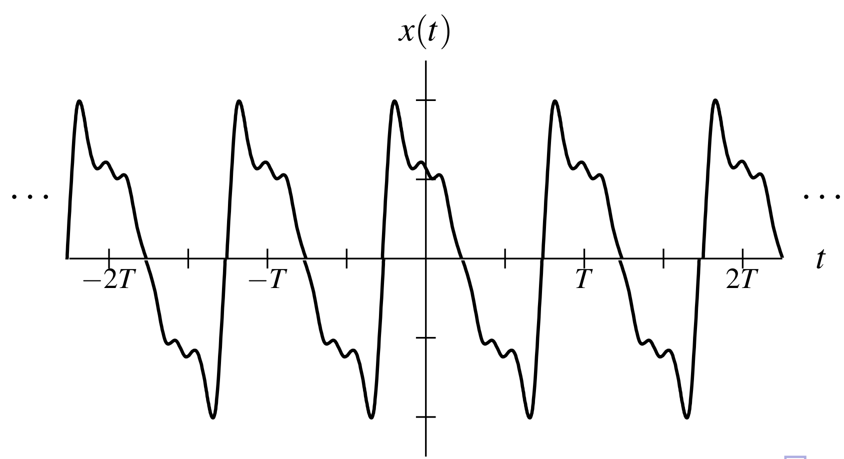
- A signal $x(t)$ is said to be **periodic** if it satisfies

$$x(t) = x(t+T), \quad \text{for all } t \text{ and some constant } T,\, T > 0.$$

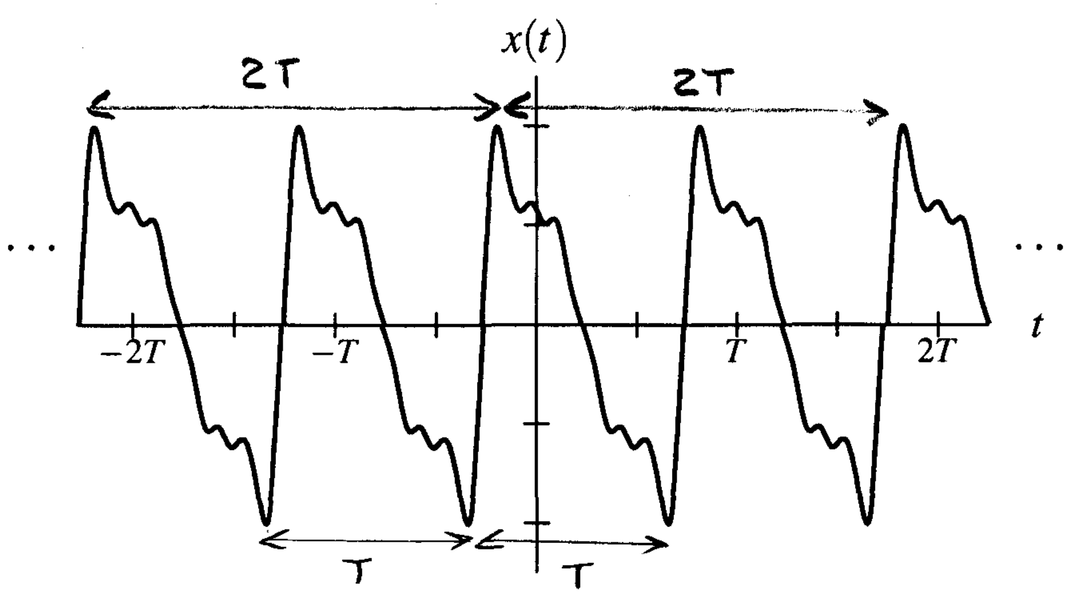
- The quantity T is referred to as the **period** of the signal.
- Two quantities closely related to the period are the **frequency** and **angular frequency**, denoted as f and ω, respectively, and defined as

$$f = \frac{1}{T} \quad \text{and} \quad \omega = 2\pi f = \frac{2\pi}{T}.$$

- A signal that is not periodic is said to be **aperiodic**.
- Examples of periodic signals include the cosine and sine functions.
- Another example of a periodic signal is shown below.



- The period of a periodic signal is ***not unique***. That is, a signal that is periodic with period T is also periodic with period NT, for every (strictly) positive integer N.



- The smallest period with which a function is periodic is called the **fundamental period** and its corresponding frequency is called the **fundamental frequency**.

Sum of Periodic Functions

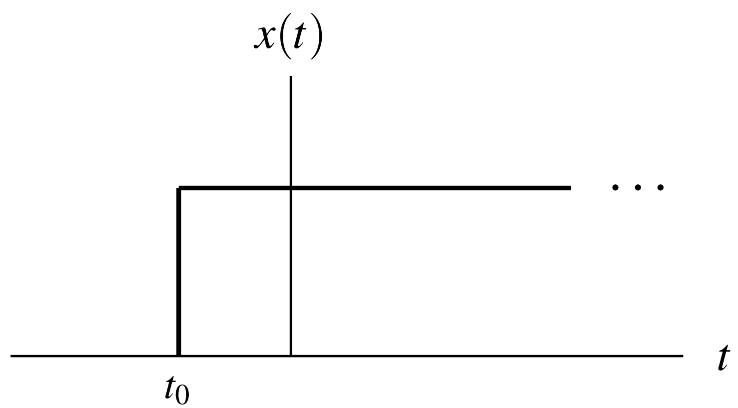
- **Sum of periodic functions.** Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2, respectively. Then, the sum $y(t) = x_1(t) + x_2(t)$ is a periodic signal if and only if the ratio T_1/T_2 ***is a rational number*** (i.e., the quotient of two integers). Suppose that $T_1/T_2 = q/r$ where q and r are integers and ***coprime*** (i.e., have no common factors), then the fundamental period of $y(t)$ is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$). (Note that rT_1 is simply the least common multiple of T_1 and T_2.)
- In passing, we note that the above result can be extended to the more general case of the sum of N periodic signals.

Right-Sided Signals

- A signal $x(t)$ is said to be **right sided** if, for some finite constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0$$

  (i.e., x is ***only nonzero to the right of*** t_0).
- An example of a right-sided signal is shown below.



- A signal $x(t)$ is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

- A causal signal is a ***special case*** of a right-sided signal.

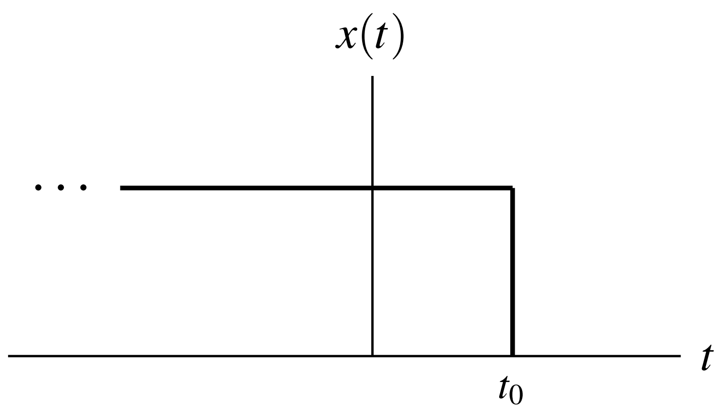
Left-Sided Signals

- A signal $x(t)$ is said to be **left sided** if, for some finite constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0$$

  (i.e., x is ***only nonzero to the left of*** t_0).

- An example of a left-sided signal is shown below.



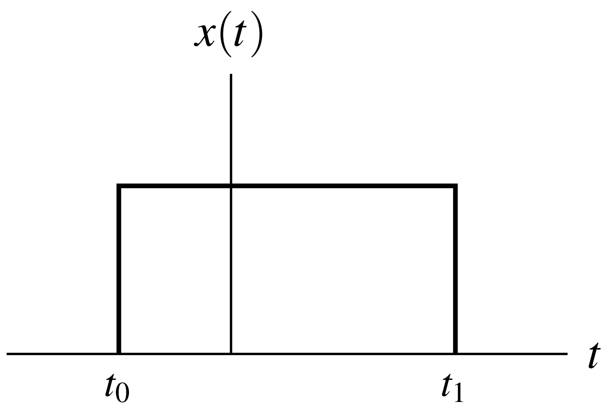
- Similarly, a signal $x(t)$ is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

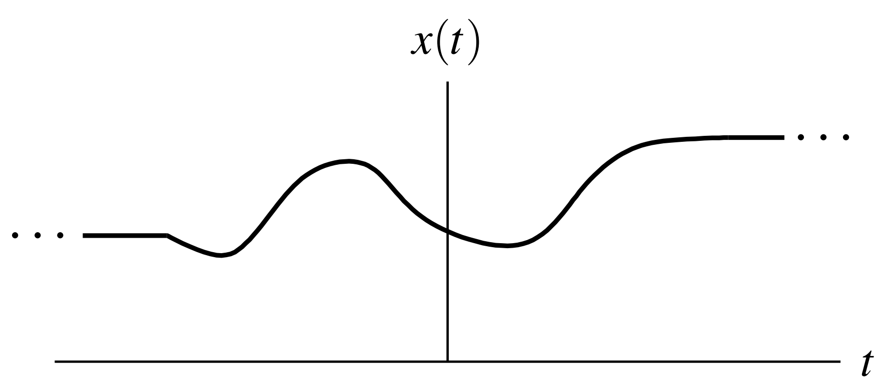
- An anticausal signal is a ***special case*** of a left-sided signal.

Finite-Duration and Two-Sided Signals

- A signal that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite duration signal is shown below.



- A signal that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided signal is shown below.



- A signal $x(t)$ is said to be **bounded** if there exists some (positive) real constant A such that

$$|x(t)| \leq A < \infty \quad \text{for all } t$$

  (i.e., $x(t)$ is ***finite*** for all t).
- Examples of bounded signals include $\sin$ and $\cos$.
- Examples of unbounded signals include $\tan$ and any (nonconstant) polynomial function.

Signal Energy and Power

- The **energy** E contained in the signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

- A signal with finite energy is said to be an **energy signal**.
- The **average power** P contained in the signal $x(t)$ is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

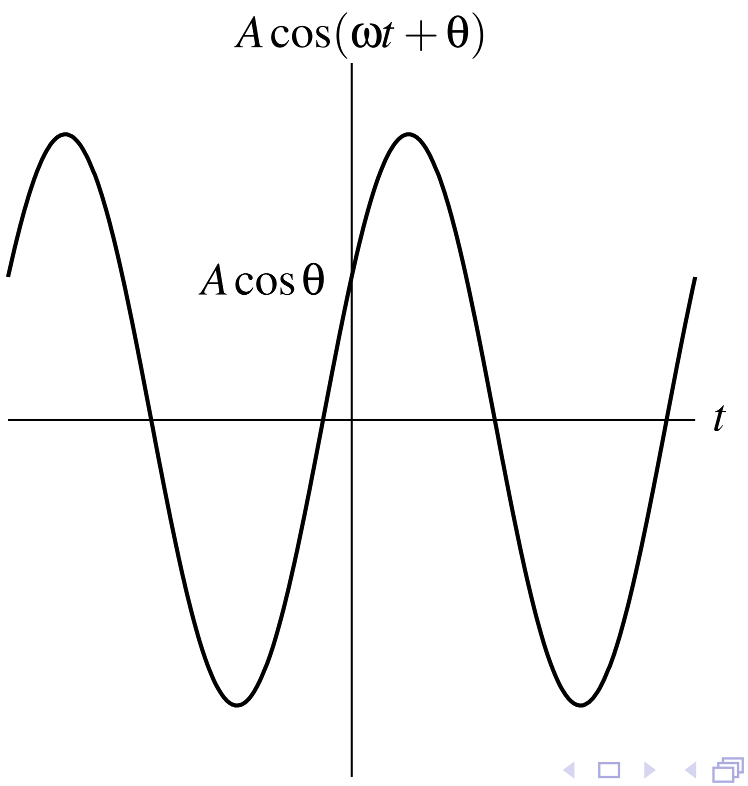
- A signal with (nonzero) finite average power is said to be a **power signal**.

Section 3.3

Elementary Signals

Real Sinusoid

- A **real sinusoid** is a function of the form
$$x(t) = A\cos(\omega t + \theta),$$
where A, ω, and θ are ***real*** constants.
- Such a function is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.
- A real sinusoid has a plot resembling that shown below.



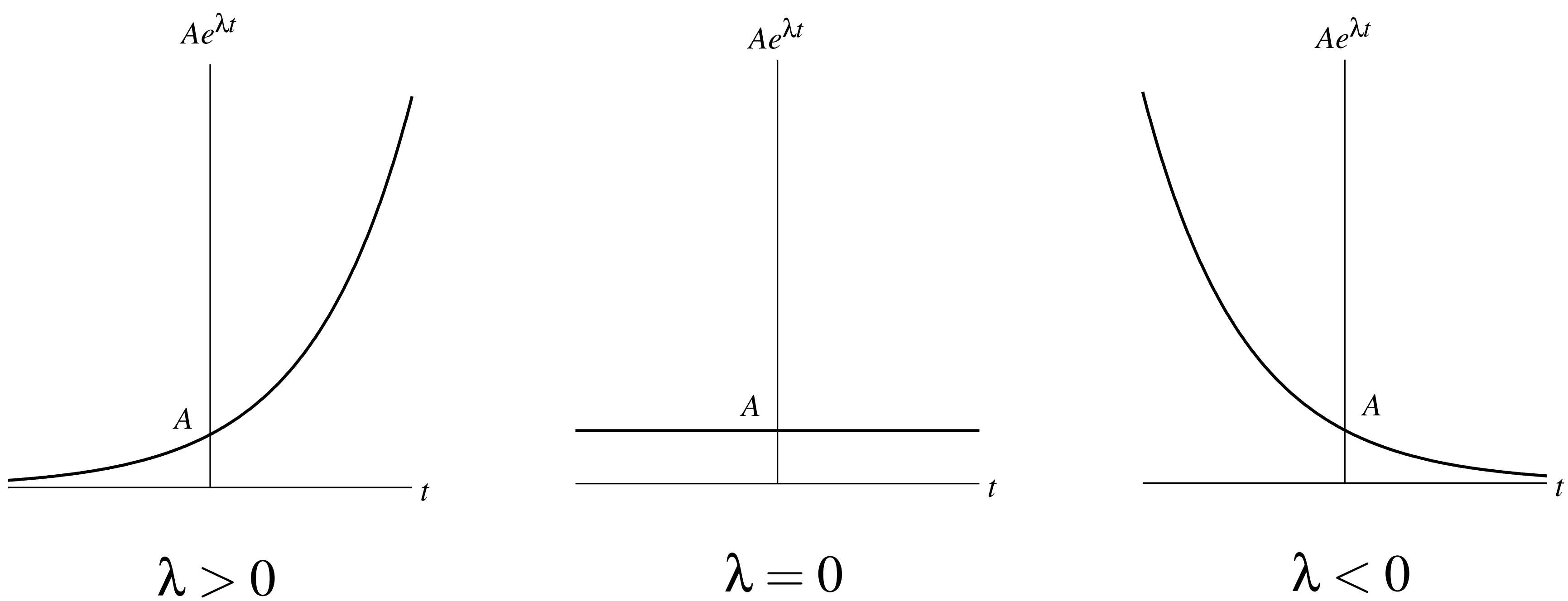
Complex Exponentials

- A **complex exponential** is a function of the form

$$x(t) = Ae^{\lambda t},$$

  where A and λ are ***complex*** constants.
- A complex exponential can exhibit one of a number of ***distinct modes of behavior***, depending on the values of its parameters A and λ.
- For example, complex exponentials include as special cases: real sinusoids, real exponentials, and complex sinusoids.

- A **real exponential** is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be ***real*** numbers.
- A real exponential can exhibit one of ***three distinct modes*** of behavior, depending on the value of λ, as illustrated below.
- If $\lambda > 0$, $x(t)$ ***increases*** exponentially as t increases (i.e., a growing exponential).
- If $\lambda < 0$, $x(t)$ ***decreases*** exponentially as t increases (i.e., a decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the ***constant*** A.



Complex Sinusoids

- A complex sinusoid is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is ***complex*** and λ is ***purely imaginary*** (i.e., $\operatorname{Re}\{\lambda\} = 0$).
- That is, a **complex sinusoid** is a function of the form

$$x(t) = Ae^{j\omega t},$$

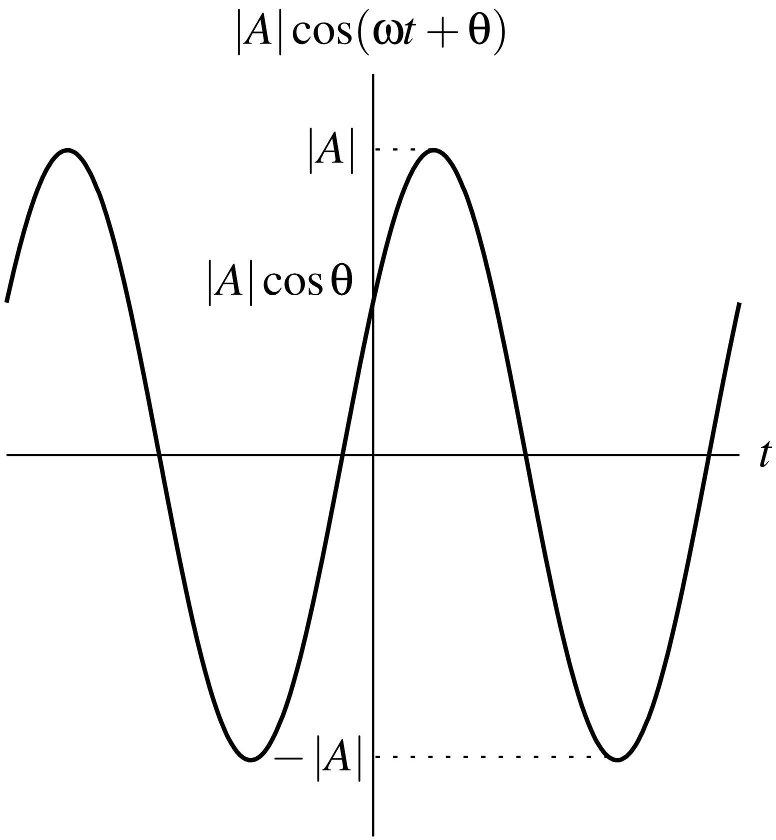
  where A is complex and ω is real.
- Letting $\theta = \arg A$ and using Euler's relation, we can rewrite $x(t)$ as

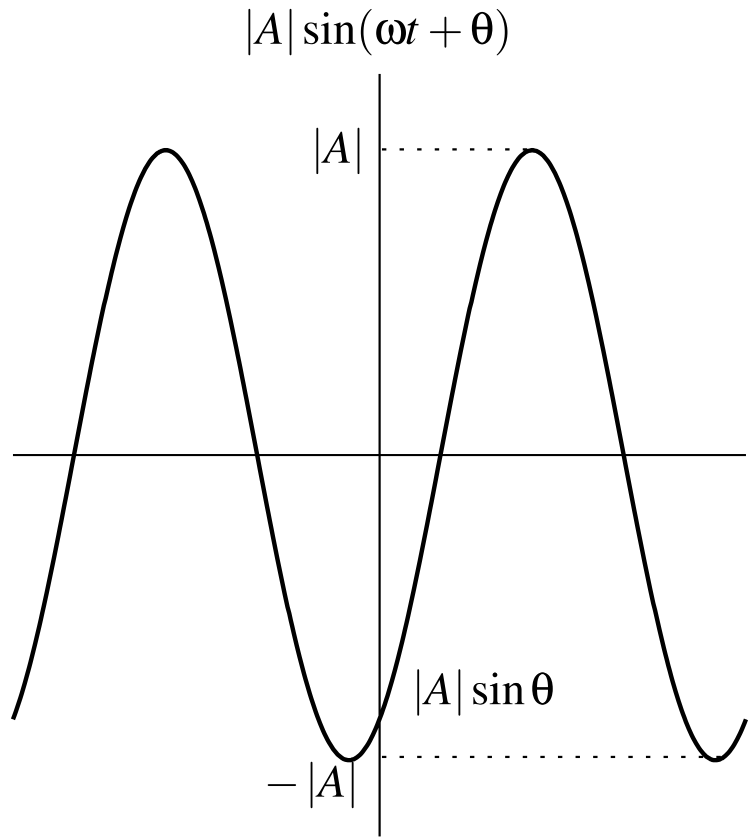
$$x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\operatorname{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\operatorname{Im}\{x(t)\}}.$$

- Thus, $\operatorname{Re}\{x(t)\}$ and $\operatorname{Im}\{x(t)\}$ are the same except for a time shift.
- Also, $x(t)$ is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.

Complex Sinusoids (Continued)

- The graphs of $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ have the forms shown below.





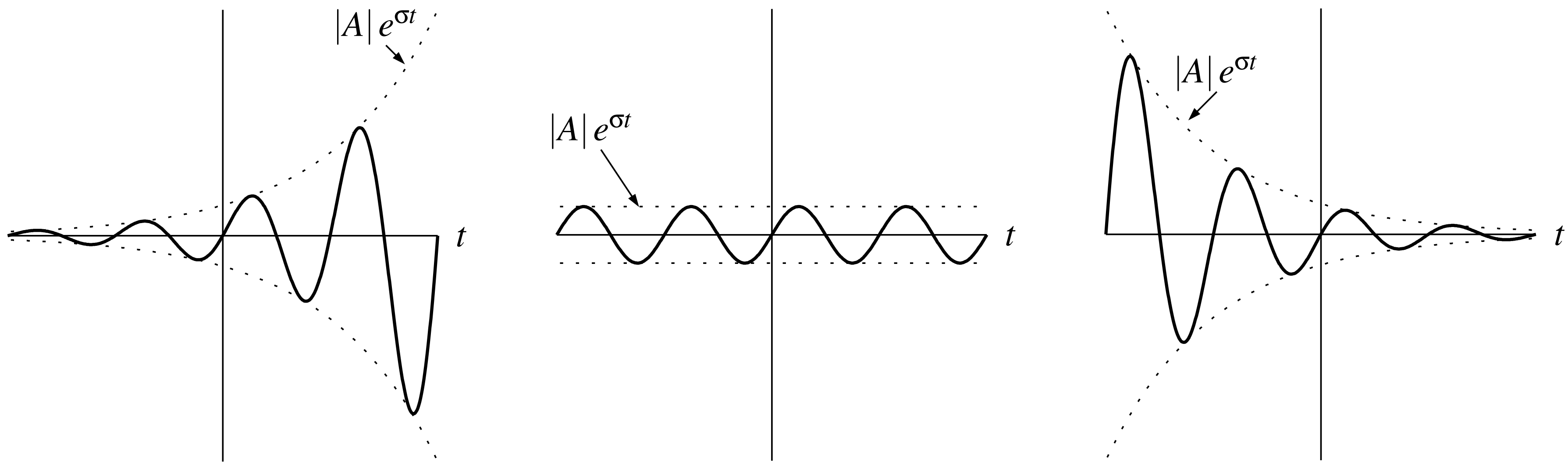
Complex Exponentials (General Case)

- In the most general case of a complex exponential $x(t) = Ae^{\lambda t}$, A and λ are both ***complex***.
- Letting $A = |A| e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A| e^{\sigma t} \cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j \underbrace{|A| e^{\sigma t} \sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are each the product of a real exponential and real sinusoid.
- One of ***three distinct modes*** of behavior is exhibited by $x(t)$, depending on the value of σ.
- If $\sigma = 0$, $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are ***real sinusoids***.
- If $\sigma > 0$, $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are each the ***product of a real sinusoid and a growing real exponential***.
- If $\sigma < 0$, $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are each the ***product of a real sinusoid and a decaying real exponential***.

- The ***three modes of behavior*** for $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are illustrated below



$\sigma > 0$ $\quad$ $\sigma = 0$ $\quad$ $\sigma < 0$

Relationship Between Complex Exponentials and Real Sinusoids

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta.$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

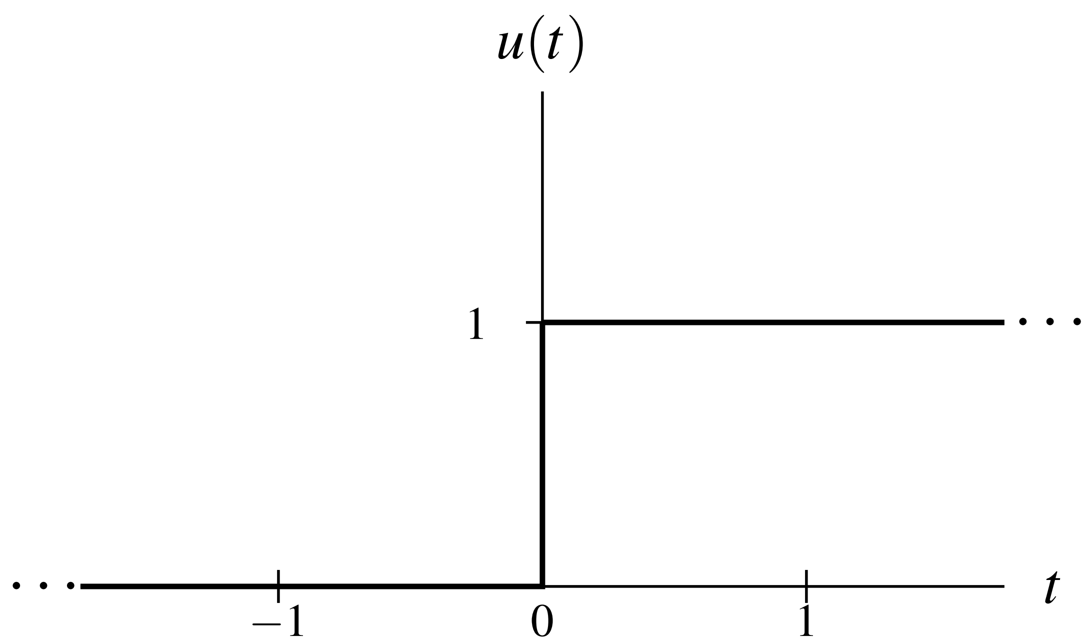
$$A\cos(\omega t + \theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \quad \text{and}$$
$$A\sin(\omega t + \theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right].$$

- Note that, above, we are simply ***restating results*** from the earlier material on complex analysis.

- The **unit-step function** (also known as the **Heaviside function**), denoted $u(t)$, is defined as

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

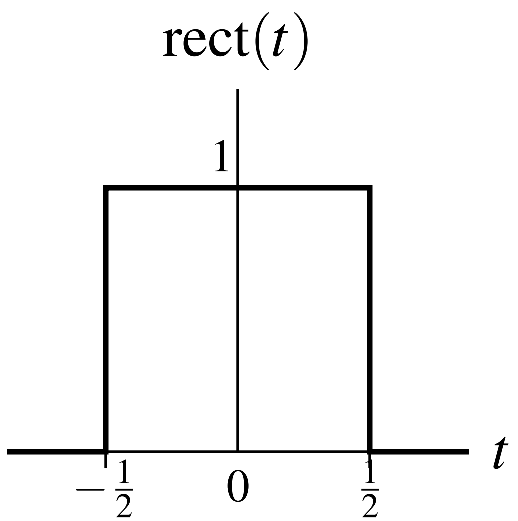
- Due to the manner in which u is used in practice, the actual ***value of*** $u(0)$ is unimportant. Sometimes values of 0 and $\frac{1}{2}$ are also used for $u(0)$.
- A plot of this function is shown below.



- The **rectangular function** (also called the unit-rectangular pulse function), denoted $\operatorname{rect}(t)$, is given by

$$\operatorname{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual ***value of*** $\operatorname{rect}(t)$ ***at*** $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.
- A plot of this function is shown below.

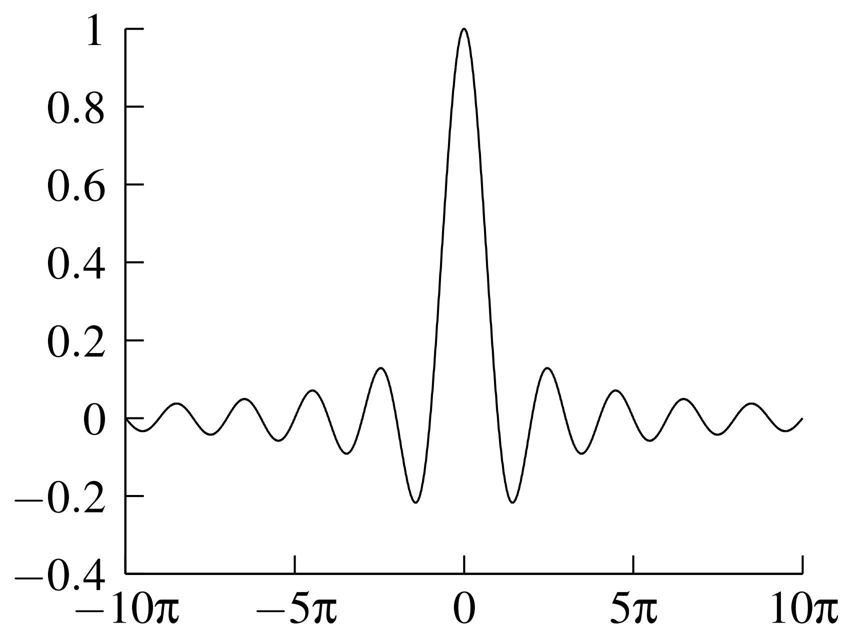


Cardinal Sine Function

- The **cardinal sine** function, denoted $\operatorname{sinc}(t)$, is given by

$$\operatorname{sinc}(t) = \frac{\sin t}{t}.$$

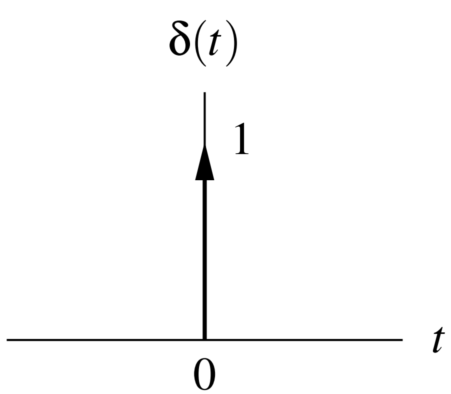
- By l'Hopital's rule, $\operatorname{sinc} 0 = 1$.
- A plot of this function for part of the real line is shown below. [Note that the oscillations in $\operatorname{sinc}(t)$ do not die out for finite t.]

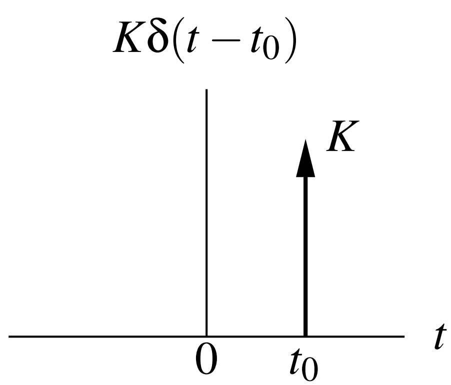


- The **unit-impulse function** (also known as the **Dirac delta function** or **delta function**), denoted $\delta(t)$, is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$
$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a ***generalized function***. Consequently, the δ function sometimes behaves in unusual ways.
- Graphically, the delta function is represented as shown below.



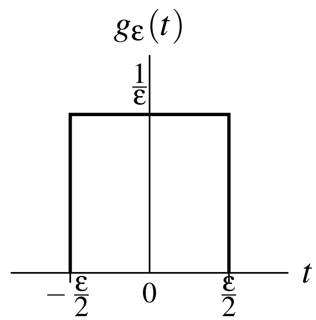


Unit-Impulse Function as a Limit

- Define

$$g_\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{for } |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$

- The function g_ε has a plot of the form shown below.



- Clearly, for any choice of ε, $\int_{-\infty}^{\infty} g_\varepsilon(t)dt = 1$.
- The function δ can be obtained as the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$

- That is, δ can be viewed as a ***limiting case of a rectangular pulse*** where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

- **Equivalence property.** For any continuous function $x(t)$ and any real constant t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

- **Sifting property.** For any continuous function $x(t)$ and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$

- The δ function also has the following properties:

$$\delta(t) = \delta(-t) \quad \text{and}$$
$$\delta(at) = \frac{1}{|a|}\delta(t),$$

where a is a nonzero real constant.

Representing a Rectangular Pulse Using Unit-Step Functions

- For real constants a and b where $a \leq b$, consider a function of the form

$$x(t) = \begin{cases} 1 & \text{if } a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

  (i.e., $x(t)$ is a ***rectangular pulse*** of height one, with a rising edge at a and falling edge at b).
- The function $x(t)$ can be equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

  (i.e., the difference of two time-shifted unit-step functions).
- Unlike the original expression for $x(t)$, this latter expression for $x(t)$ ***does not involve multiple cases***.
- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

- The idea from the previous slide can be extended to handle any function that is defined in a ***piecewise manner*** (i.e., via an expression involving multiple cases).
- That is, by using unit-step functions, we can always collapse a formula involving multiple cases into a single expression.
- Often, simplifying a formula in this way can be quite beneficial.

Section 3.4

Continuous-Time Systems

- A system with input $x(t)$ and output $y(t)$ can be described by the equation

$$y(t) = \mathcal{T}\{x(t)\},$$

where $\mathcal{T}$ denotes an operator (i.e., transformation).

- Alternatively, we can express the above relationship using the notation

$$x(t) \xrightarrow{\mathcal{T}} y(t).$$

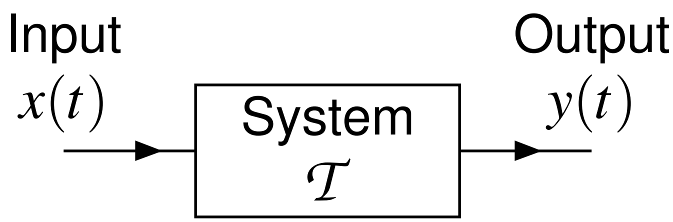
- If clear from the context, the operator $\mathcal{T}$ is often omitted, yielding the abbreviated notation

$$x(t) \to y(t).$$

- Note that the symbols "$\to$" and "=" have ***very different*** meanings.
- The symbol "$\to$" should be read as "***produces***" (not as "equals").

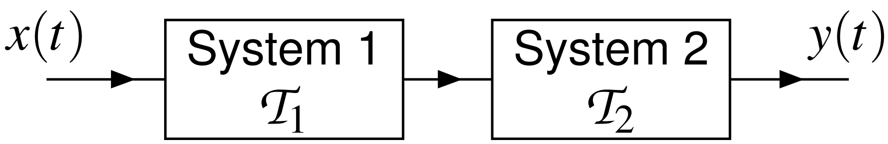
Block Diagram Representations

- Often, a system defined by the operator $\mathcal{T}$ and having the input $x(t)$ and output $y(t)$ is represented in the form of a ***block diagram*** as shown below.

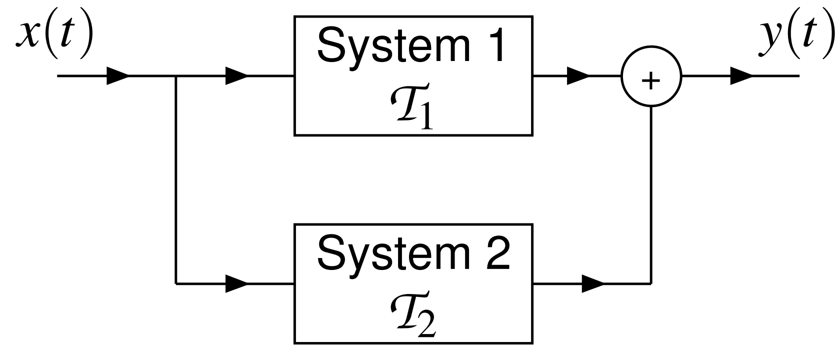


Interconnection of Systems

- *Two basic ways* in which systems can be interconnected are shown below.



Series



Parallel

- A **series** (or **cascade**) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y(t) = \mathcal{T}_2\{\mathcal{T}_1\{x(t)\}\}.$$

- A **parallel** connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y(t) = \mathcal{T}_1\{x(t)\} + \mathcal{T}_2\{x(t)\}.$$

Section 3.5

Properties of Continuous-Time Systems

- A system is said to have **memory** if its output $y(t)$ at any arbitrary time t_0 depends on the value of its input $x(t)$ at any time other than $t = t_0$.
- A system that does not have memory is said to be **memoryless**.
- Although simple, a memoryless system is ***not very flexible***, since its current output value cannot rely on past or future values of the input.
- A system is said to be **causal** if its output $y(t)$ at any arbitrary time t_0 depends only on the values of its input $x(t)$ for $t \leq t_0$.
- If the independent variable t represents time, a system must be causal in order to be ***physically realizable***.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable ***need not always represent time***. For example, in some situations, the independent variable might represent position.

Invertibility

- The **inverse** of a system $\mathcal{T}$ is another system $\mathcal{T}^{-1}$ such that the combined effect of $\mathcal{T}$ cascaded with $\mathcal{T}^{-1}$ is a system where the input and output are equal.
- A system is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input $x(t)$ can always be ***uniquely*** determined from its output $y(t)$.
- Note that the inverse/invertibility of a system and the inverse/invertibility of a function are ***fundamentally different*** things (i.e., mappings between numbers versus mappings between functions).
- An invertible system will always produce ***distinct outputs*** from any two ***distinct inputs***.
- To show that a system is ***invertible***, we simply find the ***inverse system***.
- To show that a system is ***not invertible***, we find ***two distinct inputs*** that result in ***identical outputs***.
- In practical terms, invertible systems are "nice" in the sense that their ***effects can be undone***.

Bounded-Input Bounded-Output (BIBO) Stability

- A system with input $x(t)$ and output $y(t)$ is **BIBO stable** if, for every bounded $x(t)$, $y(t)$ is bounded (i.e., $|x(t)| < \infty$ for all t implies that $|y(t)| < \infty$ for all t).
- To show that a system is ***BIBO stable***, we must show that ***every bounded input*** leads to a ***bounded output***.
- To show that a system is ***not BIBO stable***, we only need to find a single ***bounded input*** that leads to an ***unbounded output***.
- In practical terms, a BIBO stable system is ***well behaved*** in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time.
- Usually, a system that is not BIBO stable will have ***serious safety issues***. For example, an iPod with a battery input of 3.7 volts and headset output of ∞ volts = one vaporized Apple customer + one big lawsuit.

Time Invariance

- Let $y(t)$ denote the response of a system to the input $x(t)$, and let t_0 denote a real constant. If, for all $x(t)$ and all t_0, the input $x(t-t_0)$ produces the output $y(t-t_0)$, the system is said to be **time invariant (TI)**.
- That is, a system is time invariant if a ***time shift (i.e., advance or delay) in the input*** results in an ***identical time shift in the output***.
- A system that is not time invariant is said to be **time varying**.
- In simple terms, a time invariant system is a system whose behavior ***does not change*** with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much ***easier to design and analyze***, since their behavior does not change with respect to time.

- Let $y_1(t)$ and $y_2(t)$ denote the responses of a system to the inputs $x_1(t)$ and $x_2(t)$, respectively, and let a_1 and a_2 denote complex constants. If, for all $x_1(t)$, $x_2(t)$, a_1, and a_2, the input $a_1x_1(t)+a_2x_2(t)$ produces the response $a_1y_1(t)+a_2y_2(t)$, the system is said to possess the **superposition** property.
- The homogeneity property is the special case of superposition for $a_2 = 0$.
- The additivity property is the special case of superposition for $a_1 = a_2 = 1$.
- A system for which the superposition property holds is said to be **linear**.
- Practically speaking, linear systems are much ***easier to design and analyze*** than nonlinear systems. Very ***powerful tools*** have been developed for the study of linear systems. Unfortunately, equally powerful tools are not available for nonlinear systems.

Part 4

Continuous-Time Linear Time-Invariant (LTI) Systems

Why Linear Time-Invariant (LTI) Systems?

- In engineering, linear-time invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Section 4.1

Convolution

- The **convolution** of the functions $x(t)$ and $h(t)$, denoted $x(t) * h(t)$, is defined as the function

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- The convolution result $y(t) = x(t) * h(t)$ at the point t is simply the weighted average of the function $x(\tau)$, where the weighting is given by $h(-\tau)$ shifted by t.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in systems theory.
- In particular, convolution has a special significance in the context of LTI systems.

- To compute the convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau,$$

we proceed as follows:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $y(t)$.
4. Increase t gradually until the expression for $y(t)$ changes form. Record the interval over which the expression for $y(t)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.
6. The results for the various intervals can be combined in order to obtain an expression for $y(t)$ for all t.

Properties of Convolution

- The convolution operation is ***commutative***. That is, for any two functions $x(t)$ and $h(t)$,

$$x(t) * h(t) = h(t) * x(t).$$

- The convolution operation is ***associative***. That is, for any signals $x(t)$, $h_1(t)$, and $h_2(t)$,

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)].$$

- The convolution operation is ***distributive*** with respect to addition. That is, for any signals $x(t)$, $h_1(t)$, and $h_2(t)$,

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$

- For any function $x(t)$,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t).$$

- Thus, any function $x(t)$ can be written in terms of an expression involving $\delta(t)$.
- Moreover, $\delta(t)$ is the convolutional identity. That is, for any function $x(t)$,

$$x(t) * \delta(t) = x(t).$$

Section 4.2

Convolution and LTI Systems

- The response $h(t)$ of a system $\mathcal{H}$ to the input $\delta(t)$ is called the **impulse response** of the system (i.e., $\delta(t) \xrightarrow{\mathcal{H}} h(t)$).
- For any LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, the following relationship holds:

$$y(t) = x(t) * h(t).$$

- In other words, a LTI system simply computes a convolution.
- Furthermore, a LTI system is completely characterized by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.
- Since the impulse response of a LTI system is an extremely useful quantity, we often want to determine this quantity in a practical setting.
- Unfortunately, in practice, the impulse response of a system cannot be determined directly from the definition of the impulse response.

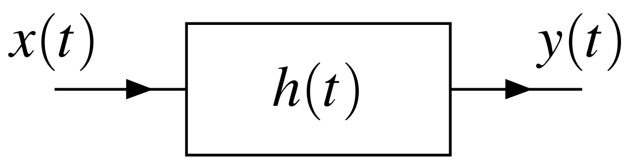
Step Response

- The response $s(t)$ of a system $\mathcal{H}$ to the input $u(t)$ is called the **step response** of the system (i.e., $u(t) \xrightarrow{\mathcal{H}} s(t)$).
- The impulse response $h(t)$ and step response $s(t)$ of a system are related as follows:

$$h(t) = \frac{ds(t)}{dt}.$$

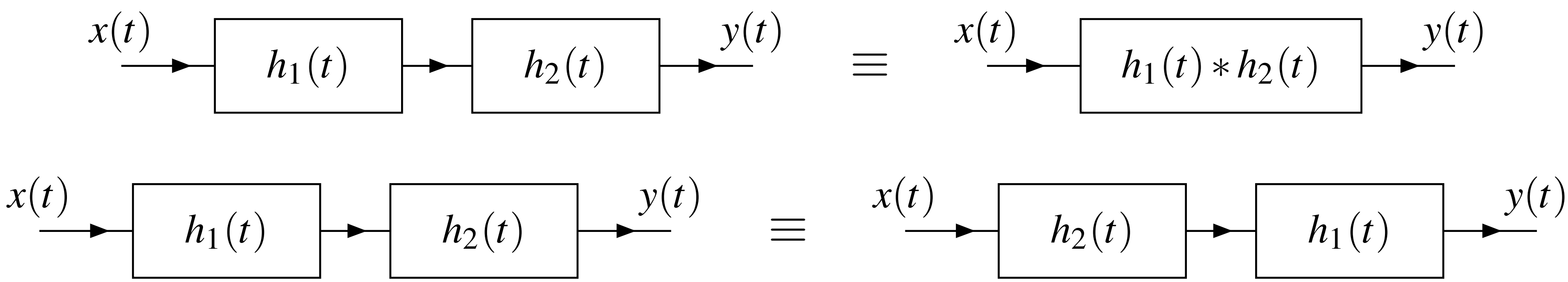
- Therefore, the impulse response of a system can be determined from its step response by differentiation.
- The step response provides a practical means for determining the impulse response of a system.

- Often, it is convenient to represent a continuous-time LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, as shown below.

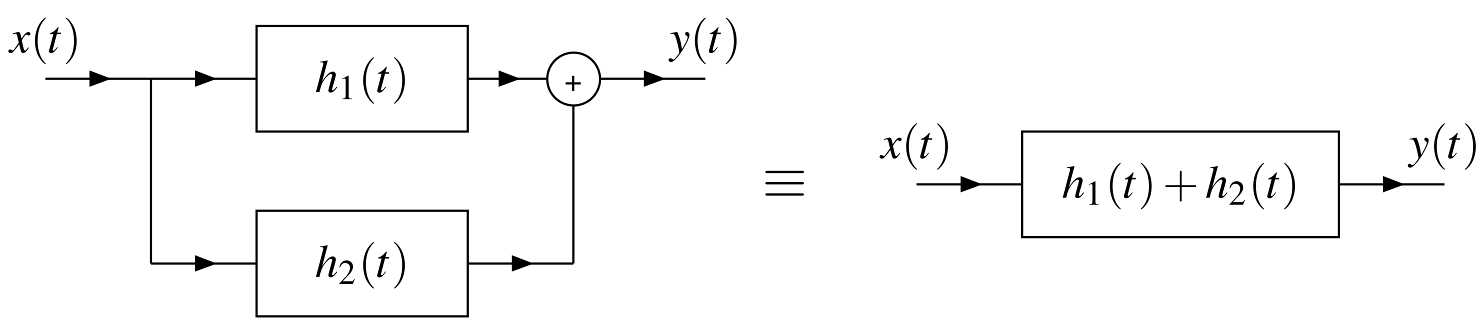


Interconnection of LTI Systems

- The ***series*** interconnection of the LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ is the LTI system with impulse response $h(t) = h_1(t) * h_2(t)$. That is, we have the equivalences shown below.



- The ***parallel*** interconnection of the LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ is a LTI system with the impulse response $h(t) = h_1(t) + h_2(t)$. That is, we have the equivalence shown below.



Section 4.3

Properties of LTI Systems

Memory

- A LTI system with impulse response $h(t)$ is memoryless if and only if

$$h(t) = 0 \quad \text{for all } t \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response $h(t)$ is of the form

$$h(t) = K\delta(t),$$

where K is a complex constant.

- Consequently, every memoryless LTI system with input $x(t)$ and output $y(t)$ is characterized by an equation of the form

$$y(t) = x(t) * K\delta(t) = Kx(t)$$

(i.e., the system is an ideal amplifier).

- For an LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

- A LTI system with impulse response $h(t)$ is causal if and only if

$$h(t) = 0 \quad \text{for } t < 0$$

(i.e., $h(t)$ is a causal signal).

- It is due to the above relationship that we call a signal $x(t)$, satisfying

$$x(t) = 0 \quad \text{for all } t < 0,$$

a causal signal.

Invertibility

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let $h(t)$ and $h^{\text{inv}}(t)$ denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h(t) * h^{\text{inv}}(t) = \delta(t).$$

- Consequently, a LTI system with impulse response $h(t)$ is invertible if and only if there exists a function $h^{\text{inv}}(t)$ such that

$$h(t) * h^{\text{inv}}(t) = \delta(t).$$

- Except in simple cases, the above condition is often quite difficult to test.

BIBO Stability

- A LTI system with impulse response $h(t)$ is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

(i.e., $h(t)$ is ***absolutely integrable***).

Eigenfunctions of Systems

- An input $x(t)$ to a system $\mathcal{H}$ is said to be an **eigenfunction** of the system $\mathcal{H}$ with the **eigenvalue** λ if the corresponding output $y(t)$ is of the form

$$y(t) = \lambda x(t),$$

  where λ is a complex constant.
- In other words, the system $\mathcal{H}$ acts as an ideal amplifier for each of its eigenfunctions $x(t)$, where the amplifier gain is given by the corresponding eigenvalue λ.
- Different systems have different eigenfunctions.
- Of particular interest are the eigenfunctions of LTI systems.

Eigenfunctions of LTI Systems

- As it turns out, every complex exponential is an eigenfunction of all LTI systems.
- For a LTI system $\mathcal{H}$ with impulse response $h(t)$,

$$e^{st} \xrightarrow{\mathcal{H}} H(s)e^{st},$$

  where s is a complex constant and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

- That is, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.
- We refer to $H(s)$ as the **system function** (or **transfer function**) of the system $\mathcal{H}$.
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$.

Representations of Signals Using Eigenfunctions

- Consider a LTI system with input $x(t)$, output $y(t)$, and system function $H(s)$.
- Suppose that the input $x(t)$ can be expressed as the linear combination of complex exponentials

$$x(t) = \sum_k a_k e^{s_k t},$$

  where the a_k and s_k are complex constants.
- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$

- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as linear combination of the ***same*** complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Part 5

Continuous-Time Fourier Series

- The Fourier series is a representation for ***periodic*** signals.
- With a Fourier series, a signal is represented as a ***linear combination of complex sinusoids***.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- For example, complex sinusoids are continuous and differentiable. They are also easy to integrate and differentiate.
- Perhaps, most importantly, complex sinusoids are ***eigenfunctions*** of LTI systems.

Section 5.1

Fourier Series

- A set of complex sinusoids is said to be **harmonically related** if there exists some constant ω_0 such that the fundamental frequency of each complex sinusoid is an integer multiple of ω_0.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(t) = e^{jk\omega_0 t} \quad \text{for } k = 0, \pm 1, \pm 2, \ldots.$$

- The fundamental frequency of the kth complex sinusoid $\phi_k(t)$ is $k\omega_0$, an integer multiple of ω_0.
- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic.
- More specifically, a linear combination of these complex sinusoids is periodic with period $T = 2\pi/\omega_0$.

- A periodic complex signal $x(t)$ with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ can be represented as a linear combination of harmonically-related complex sinusoids as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Such a representation is known as (the complex exponential form of) a **Fourier series**, and the c_k are called **Fourier series coefficients**.
- The above formula for $x(t)$ is often referred to as the **Fourier series synthesis equation**.
- The terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K\omega_0$.
- To denote that a signal $x(t)$ has the Fourier series coefficient sequence c_k, we write

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k.$$

- If a signal $x(t)$ is real, then its Fourier series coefficients c_k satisfy

$$c_k = c_{-k}^* \quad \text{for all } k.$$

- By using the above symmetry in the Fourier series coefficients and Euler's relation, the Fourier series of a real signal can be rewritten in two other forms.
- **Combined trigonometric form:**

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k),$$

where $\theta_k = \arg c_k$.
- **Trigonometric form:**

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t\right],$$

where $\alpha_k = \operatorname{Re} 2c_k$ and $\beta_k = -\operatorname{Im} 2c_k$.

- The periodic signal $x(t)$ with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ has the Fourier series coefficients c_k given by

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt,$$

where $\int_T$ denotes the integral over an arbitrary interval of length T (i.e., one period of $x(t)$).

- The above equation for c_k is often referred to as the **Fourier series analysis equation**.

Section 5.2

Convergence of Fourier Series

Convergence of Fourier Series

- Since a Fourier series can have an infinite number of terms, and an infinite sum may or may not converge, we need to consider the issue of convergence.
- That is, when we claim that a periodic signal $x(t)$ is equal to the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, is this claim actually correct?
- Consider a periodic signal $x(t)$ that we wish to represent with the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Let $x_N(t)$ denote the Fourier series truncated after the Nth harmonic components as given by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

- Here, we are interested in whether $\lim_{N\to\infty} x_N(t)$ is equal (in some sense) to $x(t)$.

Convergence of Fourier Series (Continued)

- The ***error*** in approximating $x(t)$ by $x_N(t)$ is given by

$$e_N(t) = x(t) - x_N(t),$$

  and the corresponding ***mean-squared error (MSE)*** (i.e., energy of the error) is given by

$$E_N = \frac{1}{T}\int_T |e_N(t)|^2\,dt.$$

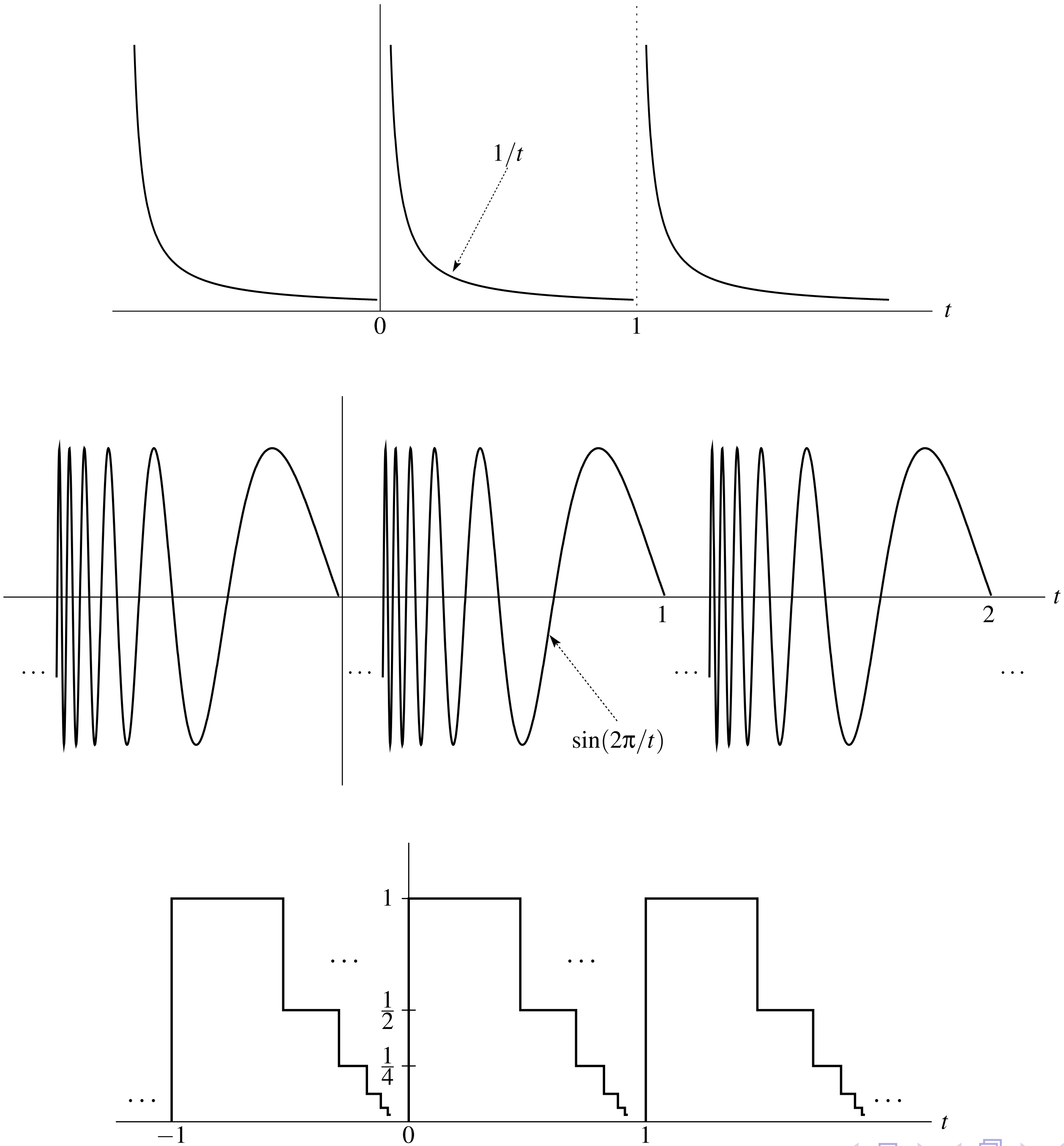
- If $\lim_{N\to\infty} e_N(t) = 0$ for all t (i.e., the error goes to zero at every point), the Fourier series is said to converge **pointwise** to $x(t)$.
- If convergence is pointwise and the rate of convergence is the same everywhere, the convergence is said to be **uniform**.
- If $\lim_{N\to\infty} E_N = 0$ (i.e., the energy of the error goes to zero), the Fourier series is said to converge to $x(t)$ in the **MSE** sense.
- Pointwise convergence implies MSE convergence, but the converse is not true. Thus, pointwise convergence is a much stronger condition than MSE convergence.

Convergence of Fourier Series: Continuous and Finite-Energy Cases

- **Continuous case.** If the periodic signal $x(t)$ is a ***continuous*** function of t, then its Fourier series converges ***uniformly*** (i.e., converges pointwise and at the same rate everywhere).
- Since, in practice, we often encounter signals with discontinuities (e.g., a square wave), the above result is of limited value.
- **Finite-energy case.** If the periodic signal $x(t)$ has ***finite energy*** in a single period (i.e., $\int_T |x(t)|^2 dt < \infty$), the Fourier series converges in the ***MSE*** sense.
- Although, in practice, most signals tend to have finite energy, MSE convergence is a somewhat weak condition. Consequently, the preceding result is not always so helpful.

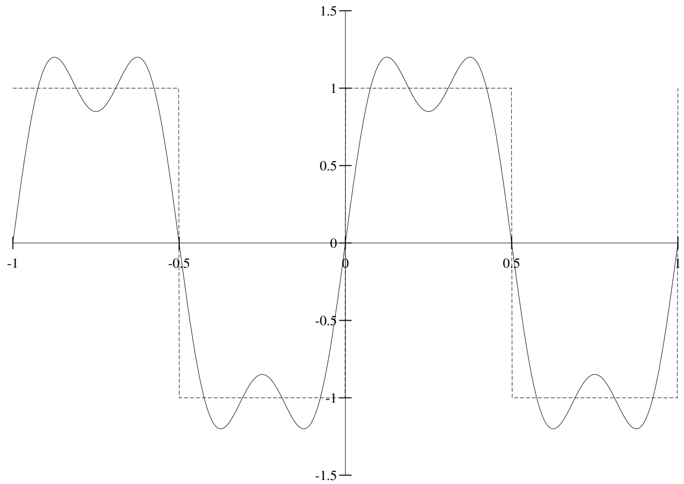
- The Dirichlet conditions for the periodic signal $x(t)$ are as follows:
  1. Over a single period, $x(t)$ is absolutely integrable (i.e., $\int_T |x(t)|\, dt < \infty$).
  2. In any finite interval of time, $x(t)$ is of bounded variation. In other words, there must be a finite number of maxima and minima in a single period of $x(t)$.
  3. In any finite interval of time, $x(t)$ has a finite number of discontinuities, each of which is finite.
- **Dirichlet case.** If $x(t)$ is a periodic signal satisfying the Dirichlet conditions, then:
  1. The Fourier series converges pointwise everywhere to $x(t)$, except at the points of discontinuity of $x(t)$.
  2. At each point $t = t_a$ of discontinuity of $x(t)$, the Fourier series converges to $\frac{1}{2}(x(t_a^-) + x(t_a^+))$ where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal on the left- and right-hand sides of the discontinuity, respectively.
- Since most signals tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier series at every point, this result is often very useful in practice.

Examples of Functions Violating the Dirichlet Conditions

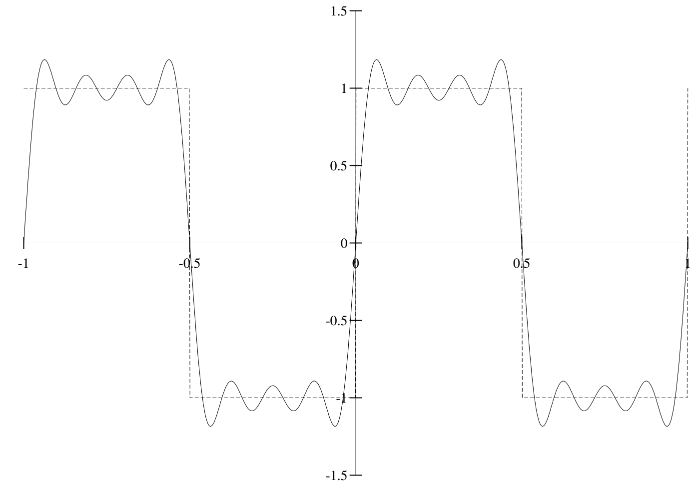


- In practice, we frequently encounter signals with discontinuities.
- When a signal $x(t)$ has discontinuities, the Fourier series representation of $x(t)$ does not converge uniformly (i.e., at the same rate everywhere).
- The rate of convergence is much slower at points in the vicinity of a discontinuity.
- Furthermore, in the vicinity of a discontinuity, the truncated Fourier series $x_N(t)$ exhibits ripples, where the peak amplitude of the ripples does not seem to decrease with increasing N.
- As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N, the peak amplitude of the ripples remains constant.
- This behavior is known as **Gibbs phenomenon**.
- The above behavior is one of the weaknesses of Fourier series (i.e., Fourier series converge very slowly near discontinuities).

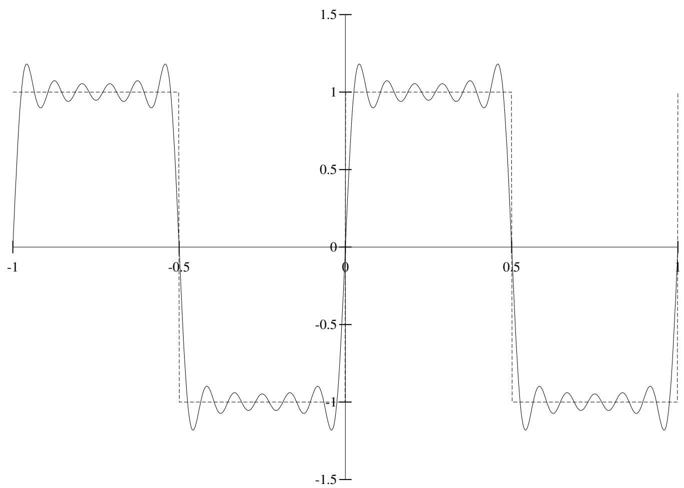
Gibbs Phenomenon: Periodic Square Wave Example



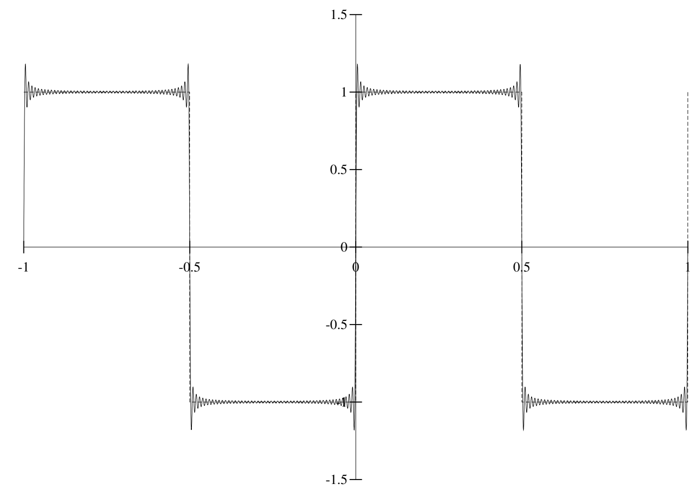
Fourier series truncated after the
3rd harmonic components



Fourier series truncated after the
7th harmonic components



Fourier series truncated after the
11th harmonic components



Fourier series truncated after the
101th harmonic components

Section 5.3

Properties of Fourier Series

- Let $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k$.
- We can show that
  - c_0 is the average value of $x(t)$ over a single period
  - $x(t)$ is real $\Leftrightarrow c_k = c_{-k}^*$ for all k
  - $x(t)$ is even $\Leftrightarrow c_k$ is even (i.e., $c_k = c_{-k}$ for all k)
  - $x(t)$ is odd $\Leftrightarrow c_k$ is odd (i.e., $c_k = -c_{-k}$ for all k)
- Using some of the preceding properties, we can infer:
  - $x(t)$ is real and even $\Leftrightarrow c_k$ is real and even
  - $x(t)$ is real and odd $\Leftrightarrow c_k$ is purely imaginary and odd
- **Linearity.** Let $x(t)$ and $y(t)$ be two periodic signals with the same period. If $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$, then

$$Ax(t) + By(t) \overset{\mathcal{FS}}{\longleftrightarrow} Aa_k + Bb_k,$$

  where A and B are complex constants.
- That is, a linear combination of signals produces the same linear combination of their Fourier series coefficients.

Properties of Fourier Series (Continued)

- **Time shifting.** Let $x(t)$ denote a periodic signal with period T and frequency $\omega_0 = 2\pi/T$. If $x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k$, then

$$x(t - t_0) \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} c_k = e^{-jk(2\pi/T)t_0} c_k,$$

  where t_0 is a real constant.
- In other words, time shifting a periodic signal changes the argument (but not magnitude) of its Fourier series coefficients.
- **Time reversal.** Let $x(t)$ denote a periodic signal with period T and frequency $\omega_0 = 2\pi/T$. If $x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k$, then

$$x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversal of a signal results in a time reversal of its Fourier series coefficients.

Section 5.4

Fourier Series and Frequency Spectra

A New Perspective on Signals: The Frequency Domain

- The Fourier series provides us with an entirely new way to view signals.
- Instead of viewing a signal as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a signal as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- This so called frequency-domain perspective is of fundamental importance in engineering.
- Many engineering problems can be solved ***much more easily*** using the frequency domain than the time domain.
- The Fourier series coefficients c_k of a signal $x(t)$ provide a means to ***quantify*** how much information $x(t)$ has at different frequencies.
- The distribution of information in a signal over different frequencies is referred to as the ***frequency spectrum*** of the signal.

- To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the signal $x(t)$, it is helpful to write the Fourier series with the c_k expressed in ***polar form*** as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} |c_k| \, e^{j(k\omega_0 t + \arg c_k)}.$$

- Clearly, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has been ***amplitude scaled*** by a factor of $|c_k|$ and ***time-shifted*** by an amount that depends on $\arg c_k$.
- For a given k, the ***larger*** $|c_k|$ is, the larger is the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the ***larger the contribution*** the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation.
- In this way, we can use $|c_k|$ as a ***measure*** of how much information a signal $x(t)$ has at the frequency $k\omega_0$.

Fourier Series and Frequency Spectra (Continued)

- The Fourier series coefficients c_k are referred to as the **frequency spectrum** of $x(t)$.
- The magnitudes $|c_k|$ of the Fourier series coefficients are referred to as the **magnitude spectrum** of $x(t)$.
- The arguments $\arg c_k$ of the Fourier series coefficients are referred to as the **phase spectrum** of $x(t)$.
- Normally, the spectrum of a signal is plotted against frequency $k\omega_0$ instead of k.
- Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, the frequency spectrum is ***discrete*** in the independent variable (i.e., frequency).
- Due to the general appearance of frequency-spectrum plot (i.e., a number of vertical lines at various frequencies), we refer to such spectra as **line spectra**.

Frequency Spectra of Real Signals

- Recall that, for a ***real*** signal $x(t)$, the Fourier series coefficient sequence c_k satisfies

$$c_k = c_{-k}^*.$$

- Trivially, from properties of complex numbers, this implies that

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}.$$

- Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a ***real*** signal is always ***even***.
- Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a ***real*** signal is always ***odd***.
- Due to the symmetry in the frequency spectra of real signals, we typically ***ignore negative frequencies*** when dealing with such signals.
- In the case of signals that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

Section 5.5

Fourier Series and LTI Systems

Frequency Response

- Recall that a LTI system $\mathcal{H}$ with impulse response $h(t)$ is such that $e^{st} \xrightarrow{\mathcal{H}} H(s)e^{st}$, where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$. (That is, complex exponentials are ***eigenfunctions*** of LTI systems.)
- Since a complex sinusoid is a ***special case*** of a complex exponential, we can reuse the above result for the special case of complex sinusoids.
- For a LTI system $\mathcal{H}$ with impulse response $h(t)$,

$$e^{j\omega t} \xrightarrow{\mathcal{H}} H(j\omega)e^{j\omega t},$$

where ω is a real constant and

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

- That is, $e^{j\omega t}$ is an ***eigenfunction*** of a LTI system and $H(j\omega)$ is the corresponding ***eigenvalue***.
- We refer to $H(j\omega)$ as the **frequency response** of the system $\mathcal{H}$.

Fourier Series and LTI Systems

- Consider a LTI system with input $x(t)$, output $y(t)$, and frequency response $H(j\omega)$.
- Suppose that the input $x(t)$ is expressed as the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Using our knowledge about the ***eigenfunctions*** of LTI systems, we can conclude

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t}.$$

- Thus, if the input $x(t)$ to a LTI system is a Fourier series, the output $y(t)$ is also a Fourier series. More specifically, if $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k$ then $y(t) \overset{\mathcal{FS}}{\longleftrightarrow} H(jk\omega_0) c_k$.
- The above formula can be used to determine the output of a LTI system from its input in a way that ***does not require convolution***.

Filtering

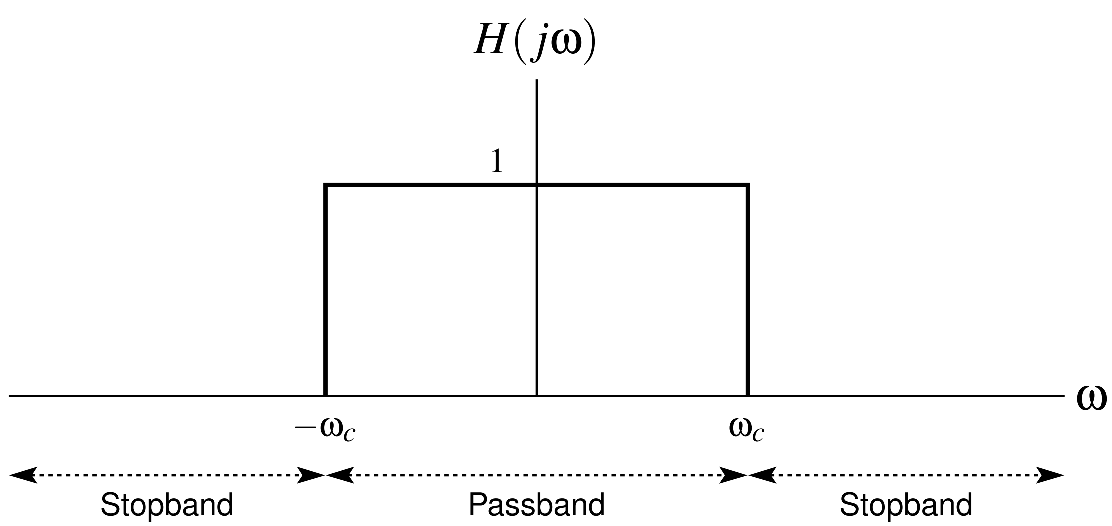
- In many applications, we want to ***modify the spectrum*** of a signal by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a signal is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

Ideal Lowpass Filter

- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

  where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

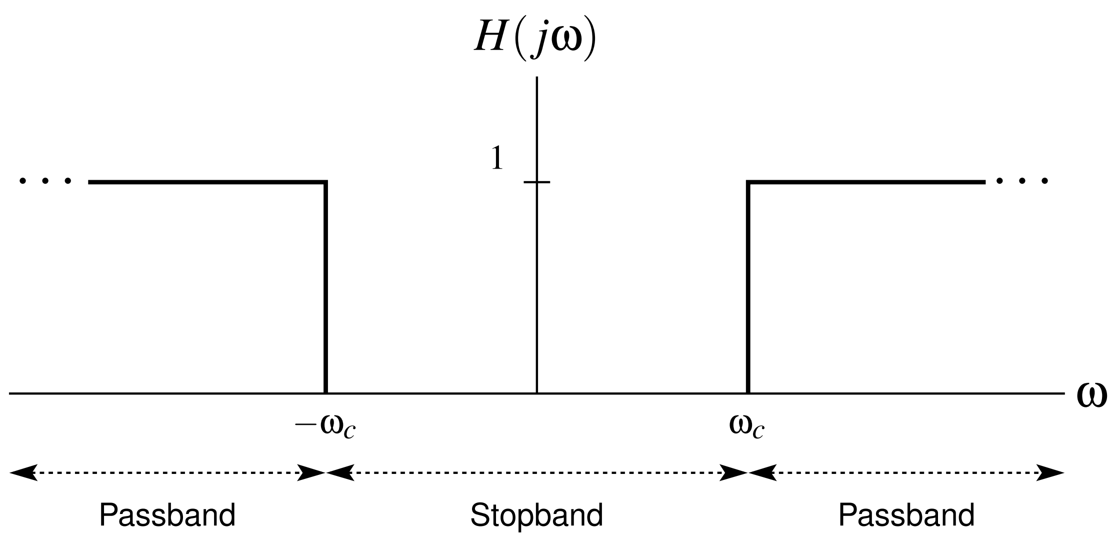


Ideal Highpass Filter

- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
  Such a filter has a ***frequency response*** of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

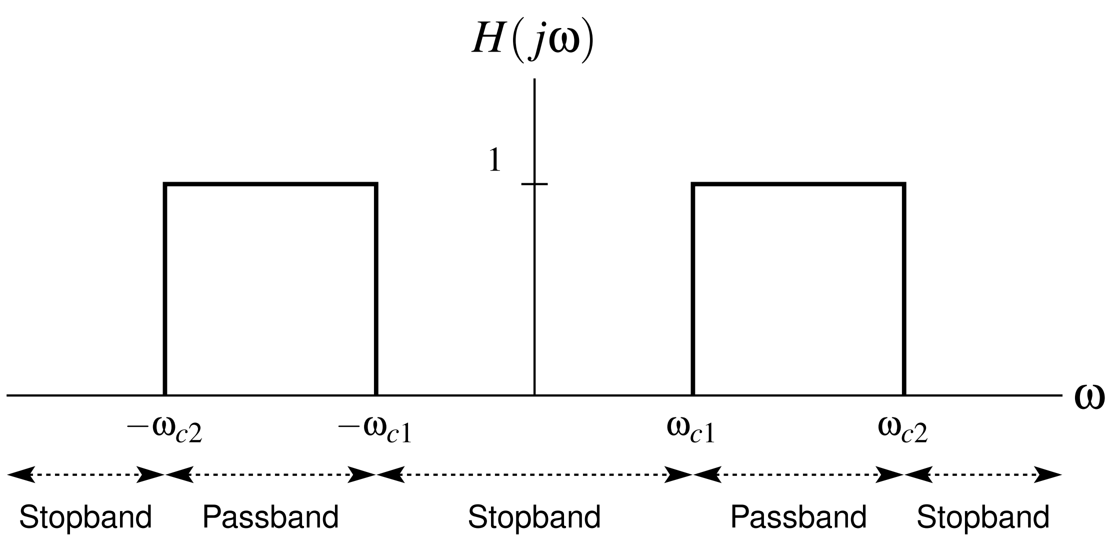
  where ω_c is the **cutoff frequency**.
  A plot of this frequency response is given below.



- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.



Part 6

Continuous-Time (CT) Fourier Transform

Motivation for Fourier Transform

- Fourier series provide an extremely useful representation for periodic signals.
- Often, however, we need to deal with signals that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The Fourier transform can be used to represent both periodic and aperiodic signals.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Section 6.1

Fourier Transform

Development of the Fourier Transform

- The Fourier series is an extremely useful signal representation.
- Unfortunately, this signal representation can only be used for periodic signals, since a Fourier series is inherently periodic.
- Many signals are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can be applied to aperiodic signals.
- By viewing an aperiodic signal as the limiting case of a periodic signal with period T where $T \to \infty$, we can use the Fourier series to develop a more general signal representation that can be used for both aperiodic and periodic signals.
- This more general signal representation is called the Fourier transform.

Fourier Transform

- The **Fourier transform** $X(\omega)$ of the signal $x(t)$, denoted $\mathcal{F}\{x(t)\}$, is given by
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$
- The preceding equation is sometimes referred to as **Fourier transform analysis equation** (or **forward Fourier transform equation**).
- Let $X(\omega)$ denote the Fourier transform of $x(t)$. Then, the **inverse Fourier transform** $x(t)$ of $X(\omega)$, denoted $\mathcal{F}^{-1}\{X(\omega)\}$, is given by
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$
- The preceding equation is sometimes referred to as the **Fourier transform synthesis equation** (or **inverse Fourier transform equation**).
- As a matter of notation, to denote that a signal $x(t)$ has the Fourier transform $X(\omega)$, we write $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$.
- A signal $x(t)$ and its Fourier transform $X(\omega)$ constitute what is called a **Fourier transform pair**.

Section 6.2

Convergence Properties of Fourier Transform

Convergence Properties of Fourier Transform

- Consider an arbitrary signal $x(t)$.
- The signal $x(t)$ has the Fourier transform representation $\hat{x}(t)$ given by

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

- Now, we need to concern ourselves with the convergence properties of this representation.
- In other words, we want to know when $\hat{x}(t)$ is a valid representation of $x(t)$.
- Since the Fourier transform is essentially derived from Fourier series, the convergence properties of the Fourier transform are closely related to the convergence properties of Fourier series.

Convergence: Finite-Energy Case

- If a signal $x(t)$ is of ***finite energy*** (i.e., $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$), then its Fourier transform representation converges in the ***MSE sense***.
- In other words, if $x(t)$ is of finite energy, then the energy E in the difference signal $\hat{x}(t) - x(t)$ is zero; that is,

$$E = \int_{-\infty}^{\infty} |\hat{x}(t) - x(t)|^2 dt = 0.$$

- It is important to note that the condition $E = 0$ does not necessarily imply $\hat{x}(t) = x(t)$ for all t.
- Thus, the above convergence result does not provide much useful information regarding the value of $\hat{x}(t)$ at specific values of t.
- Consequently, the above convergence result tends to be somewhat less useful in practice.

Convergence: Dirichlet Case

- Another important result concerning the convergence of the Fourier transform relates to what are known as the Dirichlet conditions.
- The **Dirichlet conditions** for the signal $x(t)$ are as follows:
  1. The signal $x(t)$ is ***absolutely integrable*** (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$).
  2. The signal $x(t)$ has a ***finite number of maxima and minima*** on any finite interval.
  3. The signal $x(t)$ has a ***finite number of discontinuities*** on any finite interval, and each discontinuity is itself ***finite***.
- If a signal $x(t)$ ***satisfies the Dirichlet conditions***, then its Fourier transform representation $\hat{x}(t)$ converges pointwise for all t, except at points of discontinuity. Furthermore, at each discontinuity point $t = t_a$, we have that

$$\hat{x}(t_a) = \tfrac{1}{2}\left[x(t_a^+) + x(t_a^-)\right],$$

where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal $x(t)$ on the left- and right-hand sides of the discontinuity, respectively.

Section 6.3

Properties of Fourier Transform

Properties of Fourier Transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency-Domain Convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$

Fourier Transform Pairs

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}(t/T)$	$\lvert T\rvert\operatorname{sinc}(T\omega/2)$
9	$\frac{\lvert B\rvert}{\pi}\operatorname{sinc}Bt$	$\operatorname{rect}\frac{\omega}{2B}$
10	$e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{\lvert T\rvert}{2}\operatorname{sinc}^2(T\omega/4)$

Linearity

- If $x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$, then

$$a_1x_1(t)+a_2x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} a_1X_1(\omega)+a_2X_2(\omega),$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Fourier transform.

Translation (Time-Domain Shifting)

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

  where t_0 is an arbitrary real constant.

- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

Modulation (Frequency-Domain Shifting)

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$e^{j\omega_0 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0),$$

where ω_0 is an arbitrary real constant.

- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant.

- This is known as the **dilation (or time/frequency-scaling) property** of the Fourier transform.

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then
$$x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-\omega).$$
- This is known as the **conjugation property** of the Fourier transform.

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then
$$X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$
- This is known as the **duality property** of the Fourier transform.
- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by
$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta)e^{-j\theta\lambda}d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)e^{j\theta\lambda}d\theta.$$
- That is, the forward and inverse Fourier transform equations are identical except for a ***factor of*** 2π and ***different sign*** in the parameter for the exponential function.
- Although the relationship $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform of $x(t)$, the duality property allows us to indirectly infer the Fourier transform of $X(t)$. Consequently, the duality property can be used to effectively ***double*** the number of Fourier transform pairs that we know.

- If $x_1(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t) * x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

Multiplication (Frequency-Domain Convolution)

- If $x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \tfrac{1}{2\pi}X_1(\omega) * X_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)d\theta.$$

- This is known as the **multiplication (or frequency-domain convolution) property** of the Fourier transform.
- In other words, multiplication in the time domain becomes convolution in the frequency domain (up to a scale factor of 2π).
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega).$$

- This is known as the **differentiation property** of the Fourier transform.
- Differentiation in the time domain becomes multiplication by $j\omega$ in the frequency domain.
- Of course, by repeated application of the above property, we have that $\left(\frac{d}{dt}\right)^n x(t) \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^n X(\omega)$.
- This suggests that the Fourier transform might be a useful tool when working with differential equations.

Frequency-Domain Differentiation

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then
$$tx(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega).$$
- This is known as the **frequency-domain differentiation property** of the Fourier transform.

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega).$$

- This is known as the **integration property** of the Fourier transform.
- Whereas differentiation in the time domain corresponds to ***multiplication*** by $j\omega$ in the frequency domain, integration in the time domain is associated with ***division*** by $j\omega$ in the frequency domain.
- Since integration in the time domain becomes division by $j\omega$ in the frequency domain, integration can be easier to handle in the frequency domain.

- Recall that the energy of a signal $x(t)$ is given by $\int_{-\infty}^{\infty} |x(t)|^2 dt$.
- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

- This relationship is known as **Parseval's relation**.
- That is, the energy of $x(t)$ and energy of $X(\omega)$ are equal within a scaling factor of 2π.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform ***preserves energy*** (up to a scale factor).

Fourier Transform of Periodic Signals

- Consider a periodic signal $x(t)$ with period T and frequency $\omega_0 = \frac{2\pi}{T}$.
- Define the signal $x_T(t)$ as

$$x_T(t) = \begin{cases} x(t) & \text{for } -\frac{T}{2} \le t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(i.e., $x_T(t)$ is equal to $x(t)$ over a single period and zero elsewhere).

- Let a_k denote the Fourier series coefficient sequence of $x(t)$.
- Let $X(\omega)$ and $X_T(\omega)$ denote the Fourier transforms of $x(t)$ and $x_T(t)$, respectively.
- The following relationships can be shown to hold:

$$a_k = \frac{1}{T} X_T(k\omega_0), \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0), \quad \text{and}$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0) \delta(\omega - k\omega_0).$$

Fourier Transform of Periodic Signals (Continued)

- The Fourier series coefficient sequence a_k is produced by sampling $x_T(t)$ at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$.
- The Fourier transform of a periodic signal can only be nonzero at integer multiples of the fundamental frequency.

Fourier Transform and Symmetry

- Let $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$.
- We can show that:
  - $x(t)$ is even $\Leftrightarrow X(\omega)$ is even
  - $x(t)$ is odd $\Leftrightarrow X(\omega)$ is odd

Frequency Spectra of Signals

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on signals.
- That is, instead of viewing a signal as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a signal as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- The Fourier transform $X(\omega)$ of a signal $x(t)$ provides a means to ***quantify*** how much information $x(t)$ has at different frequencies.
- The distribution of information in a signal over different frequencies is referred to as the ***frequency spectrum*** of the signal.

Fourier Transform and Frequency Spectra

- To gain further insight into the role played by the Fourier transform $X(\omega)$ in the context of the frequency spectrum of $x(t)$, it is helpful to write the Fourier transform representation of $x(t)$ with $X(\omega)$ expressed in ***polar form*** as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j[\omega t+\arg X(\omega)]}d\omega.$$

- In effect, the quantity $|X(\omega)|$ is a ***weight*** that determines how much the complex sinusoid at frequency ω contributes to the integration result $x(t)$.
- Perhaps, this can be more easily seen if we express the above integral as the ***limit of a sum***, derived from an approximation of the integral using the area of rectangles, as shown on the next slide. [Recall that $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x \to 0} \sum_{k=-\infty}^{\infty} \Delta x f(k\Delta x)$.]

Fourier Transform and Frequency Spectra (Continued 1)

- Expressing the integral (from the previous slide) as the ***limit of a sum***, we obtain

$$x(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega \left| X(\omega') \right| e^{j[\omega' t + \arg X(\omega')]},$$

  where $\omega' = k\Delta\omega$.
- From the last line of the above equation, the kth term in the summation (associated with the frequency $\omega' = k\Delta\omega$) corresponds to a complex sinusoid with fundamental frequency ω' that has had its ***amplitude scaled*** by a factor of $|X(\omega')|$ and has been ***time shifted*** by an amount that depends on $\arg X(\omega')$.
- For a given $\omega' = k\Delta\omega$ (which is associated with the kth term in the summation), the larger $|X(\omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega' t}$ will be, and therefore the larger the contribution the kth term will make to the overall summation.
- In this way, we can use $|X(\omega')|$ as a ***measure*** of how much information a signal $x(t)$ has at the frequency ω'.

Fourier Transform and Frequency Spectra (Continued 2)

- The Fourier transform $X(\omega)$ is referred to as the **frequency spectrum** of $x(t)$.
- The magnitude $|X(\omega)|$ of the Fourier transform is referred to as the **magnitude spectrum** of $x(t)$.
- The argument $\arg X(\omega)$ of the Fourier transform is referred to as the **phase spectrum** of $x(t)$.
- Since the Fourier transform is a function of a real variable, a signal can potentially have information at any real frequency.
- Earlier, we saw that for periodic signals, the Fourier transform can only be nonzero at integer multiples of the fundamental frequency.
- So, the Fourier transform and Fourier series give a consistent picture in terms of frequency spectra.
- Since the frequency spectrum is complex (in the general case), it is ***usually represented using two plots***, one showing the magnitude spectrum and one showing the phase spectrum.

Fourier Transform of Real Signals

- For a ***real*** signal $x(t)$, the Fourier transform can be shown to satisfy

$$X(\omega) = X^*(-\omega).$$

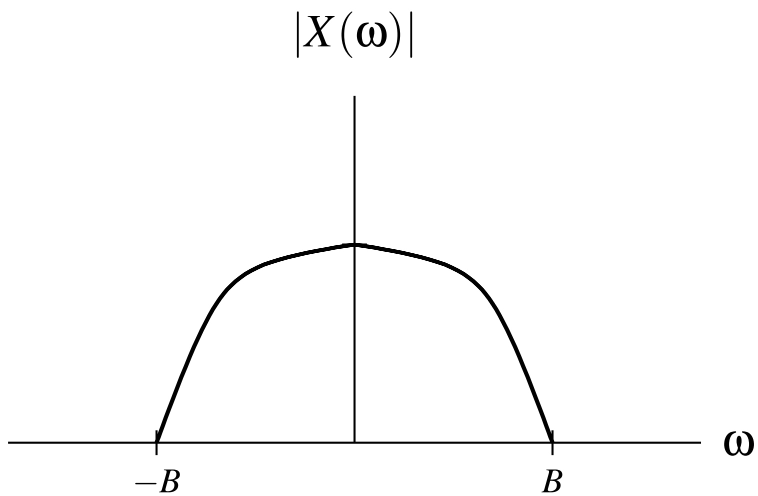
- Trivially, from properties of complex numbers, this implies that

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega).$$

- Since $|X(\omega)| = |X(-\omega)|$, the magnitude spectrum of a ***real*** signal is always ***even***.
- Similarly, since $\arg X(\omega) = -\arg X(-\omega)$, the phase spectrum of a ***real*** signal is always ***odd***.
- Due to the symmetry in the frequency spectra of real signals, we typically ***ignore negative frequencies*** when dealing with such signals.
- In the case of signals that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

Bandwidth

- A signal $x(t)$ with Fourier transform $X(\omega)$ is said to be **bandlimited** if, for some nonnegative real constant B, $X(\omega) = 0$ for all ω satisfying $|\omega| > B$.
- In the context of real signals, we usually refer to B as the **bandwidth** of the signal $x(t)$.
- The signal with the Fourier transform $X(\omega)$ shown below has bandwidth B.



- One can show that a signal ***cannot be both time limited and bandlimited***. (This follows from the time/frequency scaling property of the Fourier transform.)

Section 6.4

Fourier Transform and LTI Systems

Frequency Response of LTI Systems

- Consider a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Let $X(\omega)$, $Y(\omega)$, and $H(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.
- Since $y(t) = x(t) * h(t)$, we have that

$$Y(\omega) = X(\omega)H(\omega).$$

- The preceding equation provides an alternative way of viewing the behavior of an LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output signals.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency spectrum of the impulse response.
- As a matter of terminology, we refer to $H(\omega)$ as the **frequency response** of the system.
- A LTI system is completely characterized by its frequency response $H(\omega)$.

Frequency Response of LTI Systems (Continued 1)

- In the general case, the frequency response $H(\omega)$ is a complex-valued function.
- Often, we represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$.
- We call $|H(\omega)|$ the **magnitude response** of the system.
- We call $\arg H(\omega)$ the **phase response** of the system.
- Since $Y(\omega) = X(\omega)H(\omega)$, we trivially have that

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)| \quad \text{and} \quad \arg Y(\omega) = \arg X(\omega) + \arg H(\omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude spectrum of the impulse response.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase spectrum of the impulse response.

Frequency Response of LTI Systems (Continued 2)

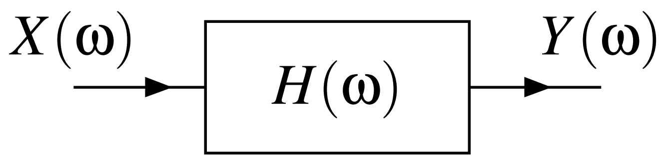
- Since the frequency response $H(\omega)$ is simply the frequency spectrum of the impulse response $h(t)$, if $h(t)$ is ***real***, then

$$|H(\omega)| = |H(-\omega)| \quad \text{and} \quad \arg H(\omega) = -\arg H(-\omega)$$

(i.e., the magnitude response $|H(\omega)|$ is ***even*** and the phase response $\arg H(\omega)$ is ***odd***).

Block Diagram Representations of LTI Systems

- Consider a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$.
- Let $X(\omega)$, $Y(\omega)$, and $H(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.
- Often, it is convenient to represent such a system in block diagram form in the frequency domain as shown below.



- Since a LTI system is completely characterized by its frequency response, we typically label the system with this quantity.

Frequency Response and Differential Equation Representations of LTI Systems

- Many LTI systems of practical interest can be represented using an ***Nth-order linear differential equation with constant coefficients***.
- Consider a system with input $x(t)$ and output $y(t)$ that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t) \quad \text{where} \quad M \leq N.$$

- Let $h(t)$ denote the impulse response of the system, and let $X(\omega)$, $Y(\omega)$, and $H(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.
- One can show that $H(\omega)$ is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

- Observe that, for a system of the form considered above, the frequency response is a ***rational function***.

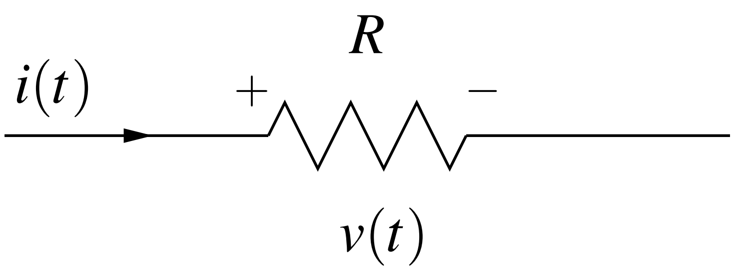
Section 6.5

Application: Circuit Analysis

- A **resistor** is a circuit element that opposes the flow of electric current.
- A resistor is governed by the relationship

$$v(t) = Ri(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad V(\omega) = RI(\omega)$$

  where R, $v(t)$ and $i(t)$ denote the resistance of, voltage across, and current through the resistor, respectively.
- In circuit diagrams, a resistor is denoted by the symbol shown below.



Inductors

- An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa.
- The inductor is governed by the relationship

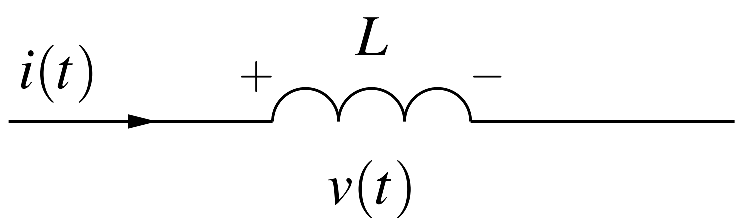
$$v(t) = L\tfrac{d}{dt}i(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad V(\omega) = j\omega L I(\omega)$$

or equivalently

$$i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad I(\omega) = \tfrac{1}{j\omega L}V(\omega)$$

where L, $v(t)$, and $i(t)$ denote the inductance of, voltage across, and current through the inductor, respectively.

- In circuit diagrams, a inductor is denoted by the symbol shown below.



- A **capacitor** is a circuit element that stores electric charge.
- The capacitor is governed by the relationship

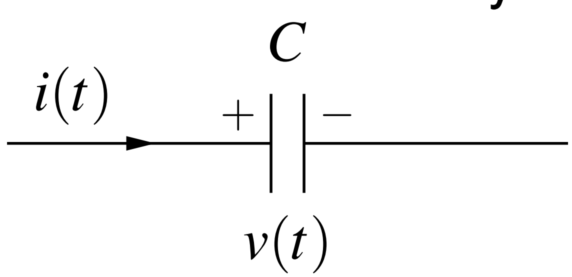
$$v(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad V(\omega) = \tfrac{1}{j\omega C}I(\omega)$$

or equivalently

$$i(t) = C\tfrac{d}{dt}v(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad I(\omega) = j\omega C V(\omega)$$

where C, $v(t)$, and $i(t)$ denote the capacitance of, voltage across, and current through the capacitor, respectively.

- In circuit diagrams, a capacitor is denoted by the symbol shown below.



- The Fourier transform is very useful tool for circuit analysis.
- The utility of the Fourier transform is partly due to the fact that the ***differential/integral*** equations that describe inductors and capacitors are much simpler to express in the Fourier domain than in the time domain.

Section 6.6

Application: Filtering

Filtering

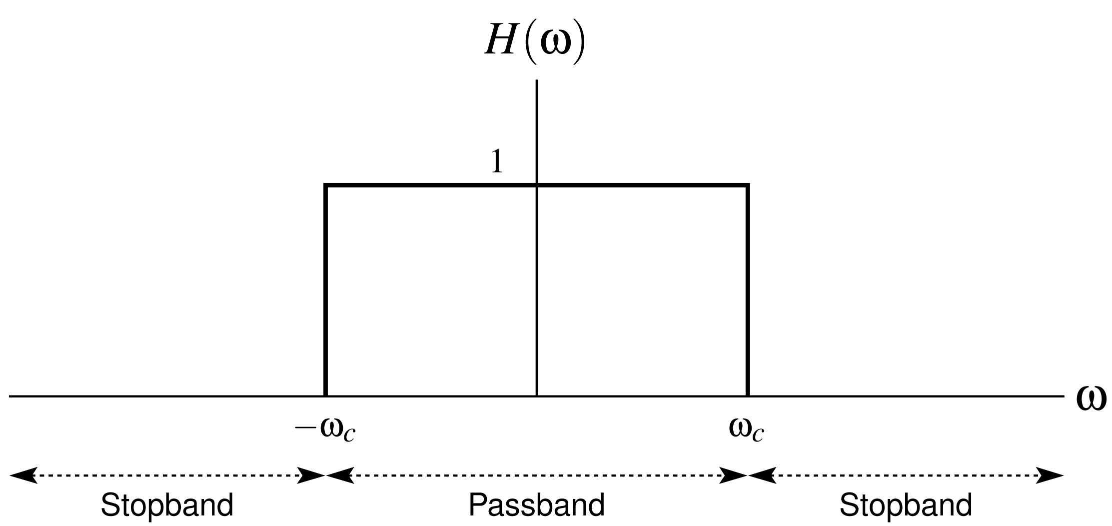
- In many applications, we want to ***modify the spectrum*** of a signal by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a signal is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

Ideal Lowpass Filter

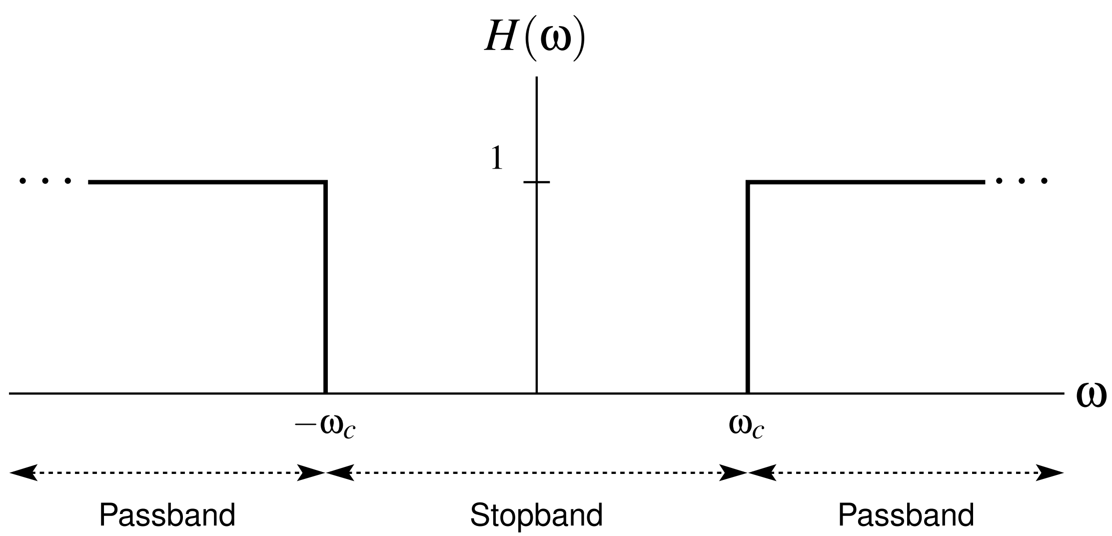
- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

  where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.



- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
  Such a filter has a ***frequency response*** of the form
  $$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$
  where ω_c is the **cutoff frequency**.
  A plot of this frequency response is given below.

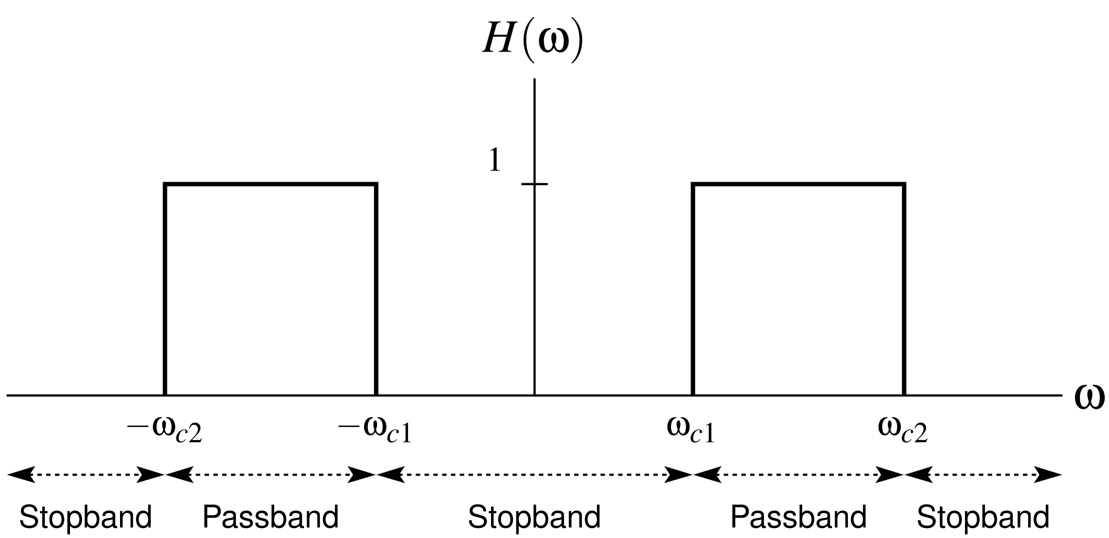


Ideal Bandpass Filter

- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(\omega) = \begin{cases} 1 & \text{for } \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.

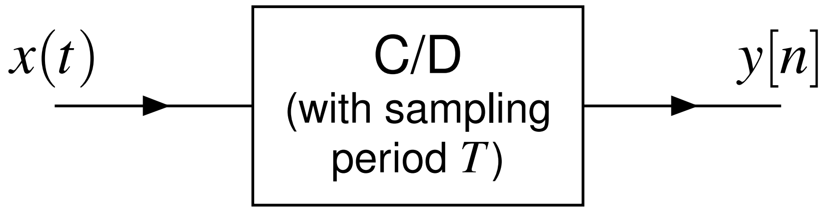


Section 6.7

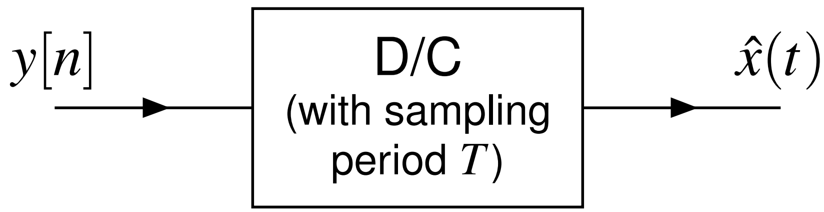
Application: Sampling and Interpolation

Sampling and Interpolation

- Often, we want to be able to ***convert*** between continuous-time and discrete-time representations of a signal.
- This is accomplished through processes known as ***sampling*** and ***interpolation***.
- The ***sampling*** process, which is performed by an **ideal continuous-time to discrete-time (C/D) converter** shown below, transforms a continuous-time signal $x(t)$ to a discrete-time signal (i.e., sequence) $y[n]$.



- The ***interpolation*** process, which is performed by an **ideal discrete-time to continuous-time (D/C) converter** shown below, transforms a discrete-time signal $y[n]$ to a continuous-time signal $x(t)$.

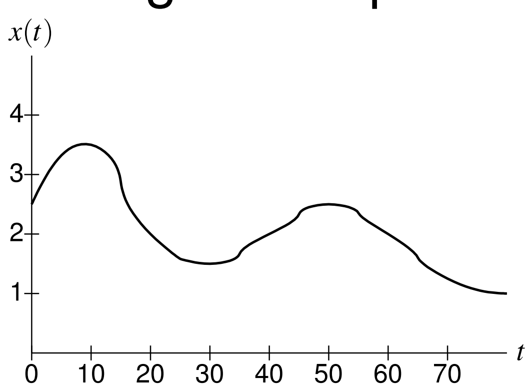


- Note that, unless very special conditions are met, the sampling process loses information (i.e., is ***not invertible***).

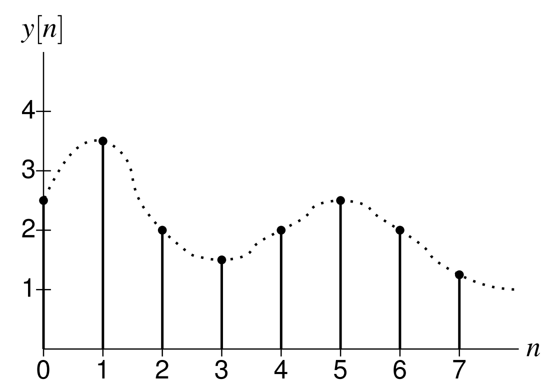
- Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**.
- With this scheme, a sequence $y[n]$ of samples is obtained from a continuous-time signal $x(t)$ according to the relation

$$y[n] = x(nT) \quad \text{for all integer } n,$$

where T is a positive real constant.
- As a matter of terminology, we refer to T as the **sampling period**, and $\omega_s = 2\pi/T$ as the (angular) **sampling frequency**.
- An example of periodic sampling is shown below, where the original continuous-time signal $x(t)$ has been sampled with ***sampling period*** $T = 10$, yielding the sequence $y[n]$.



Original Signal



Sampled Signal

Periodic Sampling (Continued)

- The sampling process is not generally invertible.
- In the absence of any constraints, a continuous-time signal cannot usually be uniquely determined from a sequence of its equally-spaced samples.
- Consider, for example, the continuous-time signals $x_1(t)$ and $x_2(t)$ given by

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

If we sample each of these signals with the sampling period $T = 1$, we obtain the respective sequences

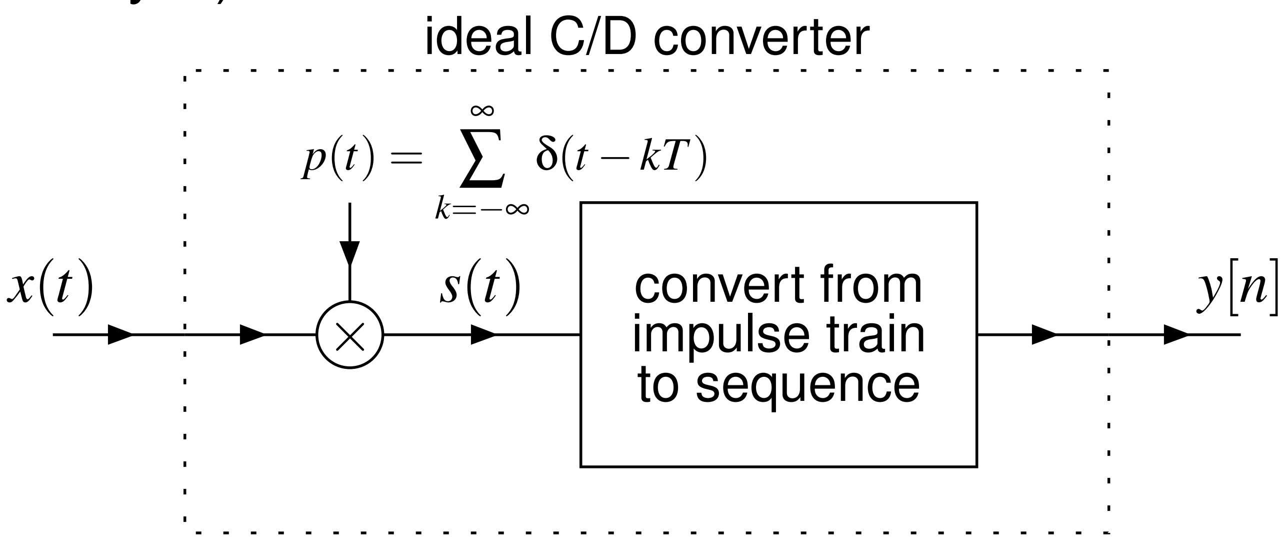
$$\begin{aligned} y_1[n] &= x_1(nT) = x_1(n) = 0 \quad \text{and} \\ y_2[n] &= x_2(nT) = \sin(2\pi n) = 0. \end{aligned}$$

Thus, $y_1[n] = y_2[n]$ for all n, although $x_1(t) \neq x_2(t)$ for all noninteger t.

- Fortunately, under certain circumstances, a continuous-time signal can be recovered exactly from its samples.

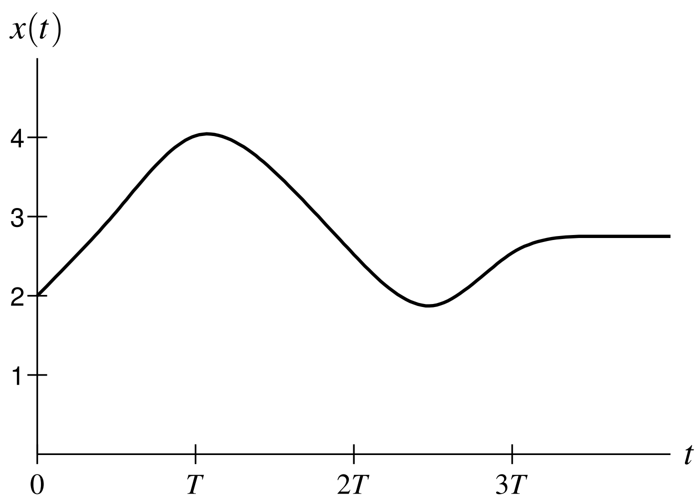
Model of Sampling

- For the purposes of analysis, sampling with sampling period T and frequency $\omega_s = \frac{2\pi}{T}$ can be modelled as shown below.
- An **impulse train** is a signal of the form $v(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - kT)$, where a_k and T are real constants (i.e., $v(t)$ consists of weighted impulses spaced apart by T).

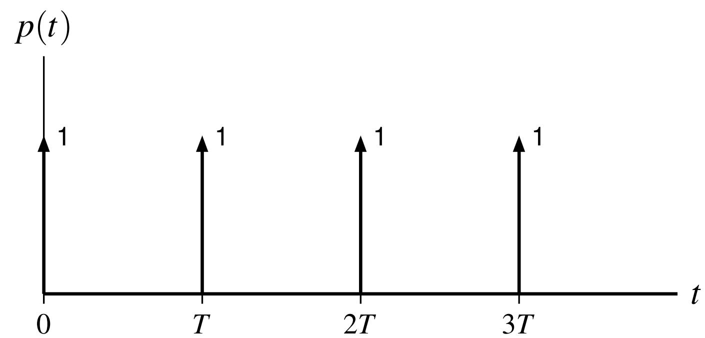


- The sampling of a continuous-time signal $x(t)$ to produce a sequence $y[n]$ consists of the following two steps (in order):
  1. Multiply the signal $x(t)$ to be sampled by a periodic impulse train $p(t)$, yielding the impulse train $s(t)$.
  2. Convert the impulse train $s(t)$ to a sequence $y[n]$ (by forming a sequence from the weights of successive impulses in the impulse train).

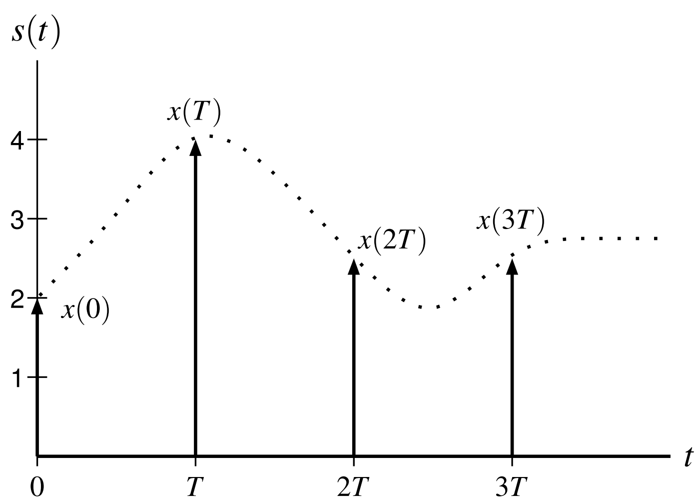
Model of Sampling: Various Signals



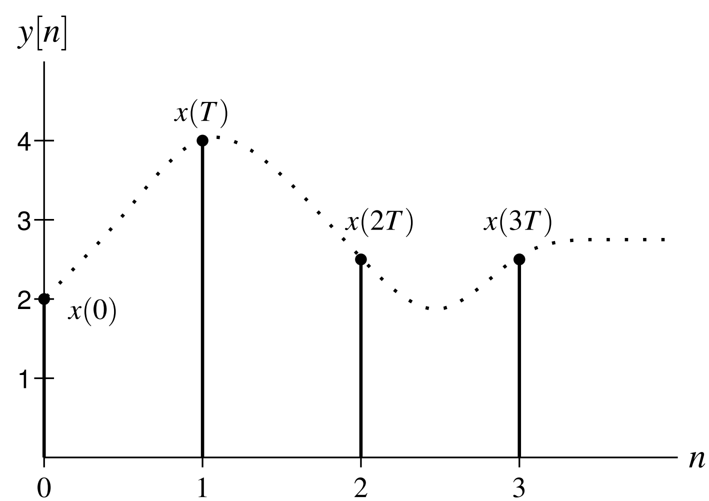
Input Signal (Continuous-Time)



Periodic Impulse Train

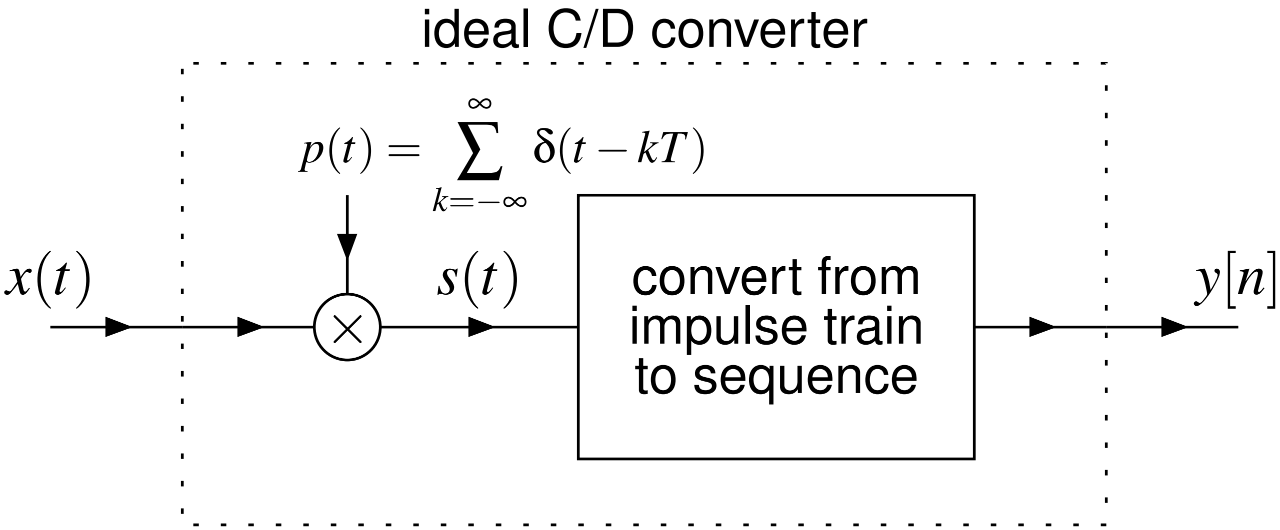


Impulse-Sampled Signal *(Continuous-Time)*



Output Sequence *(Discrete-Time)*

Model of Sampling: Characterization



- In the time domain, the impulse-sampled signal $s(t)$ is given by

$$s(t) = x(t)p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

- In the Fourier domain, the preceding equation becomes

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

- Thus, the spectrum of the impulse-sampled signal $s(t)$ is a scaled sum of an infinite number of ***shifted copies*** of the spectrum of the original signal $x(t)$.

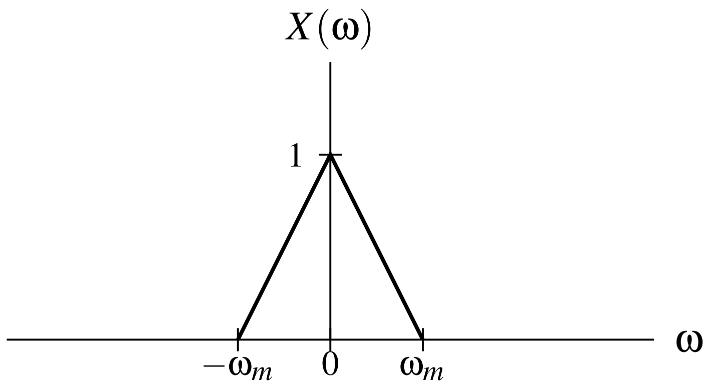
Model of Sampling: Aliasing

- Consider frequency spectrum of the impulse-sampled signal $s(t)$ given by

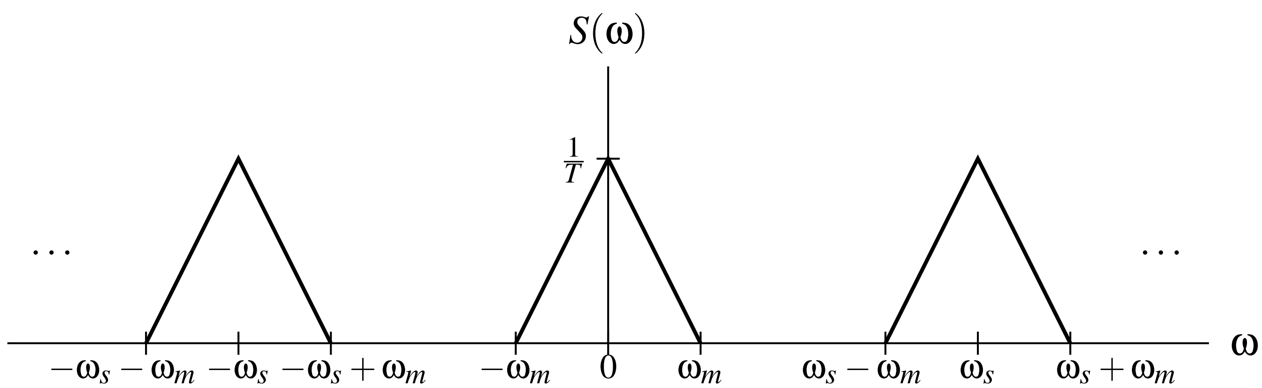
$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

- The function $S(\omega)$ is a scaled sum of an infinite number of ***shifted copies*** of $X(\omega)$.
- Two distinct behaviors can result in this summation, depending on ω_s and the bandwidth of $x(t)$.
- In particular, the nonzero portions of the different shifted copies of $X(\omega)$ can either:
  1. overlap; or
  2. not overlap.
- In the case where overlap occurs, the various shifted copies of $X(\omega)$ add together in such a way that the original shape of $X(\omega)$ is lost. This phenomenon is known as **aliasing**.
- When aliasing occurs, the original signal $x(t)$ cannot be recovered from its samples $y[n]$.

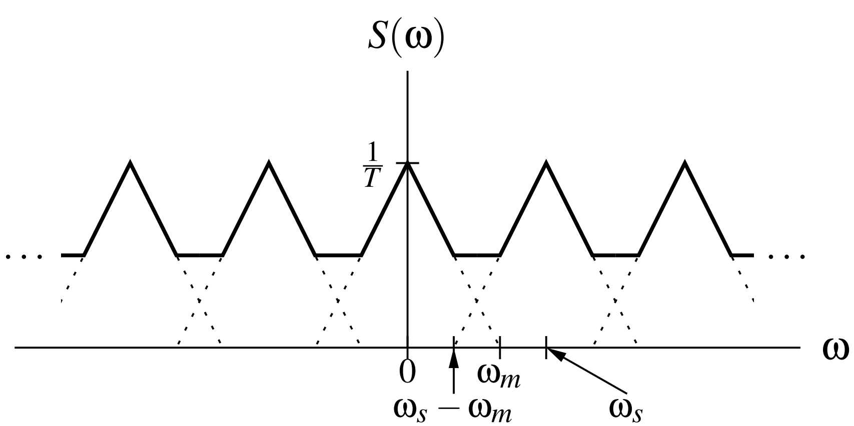
Model of Sampling: Aliasing (Continued)



Spectrum of Input Signal
(Bandwidth ω_m)

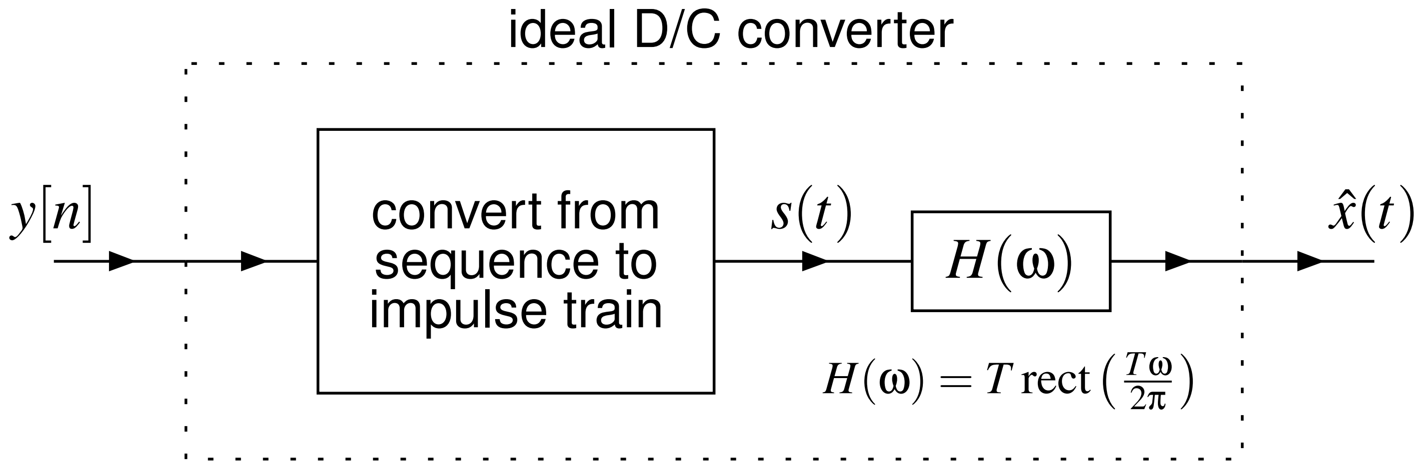


Spectrum of Impulse-Sampled Signal:
No Aliasing Case
($\omega_s > 2\omega_m$)



Spectrum of Impulse-Sampled Signal:
Aliasing Case
($\omega_s \le 2\omega_m$)

- For the purposes of analysis, interpolation can be modelled as shown below.



- The inverse Fourier transform of $H(\omega)$ is $h(t) = \operatorname{sinc}(\pi t/T)$.
- The reconstruction of a continuous-time signal $x(t)$ from its samples $y[n]$ (i.e., bandlimited interpolation) consists of the following two steps (in order):
  1. Convert the sequence $y[n]$ to the impulse train $s(t)$ (by using the elements in the sequence as the weights of successive impulses in the impulse train).
  2. Apply a lowpass filter to $s(t)$ to produce $\hat{x}(t)$.
- The lowpass filter is used to eliminate the extra copies of the original signal's spectrum present in the spectrum of the impulse-sampled signal $s(t)$.

- In more detail, the reconstruction process proceeds as follows.
- First, we convert the sequence $y[n]$ to the impulse train $s(t)$ to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y[n]\delta(t - nT).$$

- Then, we filter the resulting signal $s(t)$ with the lowpass filter having impulse response $h(t)$, yielding

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} y[n]\operatorname{sinc}\left(\tfrac{\pi}{T}(t - nT)\right).$$

- **Sampling Theorem.** Let $x(t)$ be a signal with Fourier transform $X(\omega)$, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., $x(t)$ is bandlimited to the interval $[-\omega_M, \omega_M]$). Then, $x(t)$ is uniquely determined by its samples $y[n] = x(nT)$ for all integer n, if

$$\omega_s > 2\omega_M,$$

where $\omega_s = 2\pi/T$. The preceding inequality is known as the **Nyquist condition**. If this condition is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}(\tfrac{\pi}{T}(t - nT)),$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}(\tfrac{\omega_s}{2}t - \pi n).$$

- We call $\omega_s/2$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**.

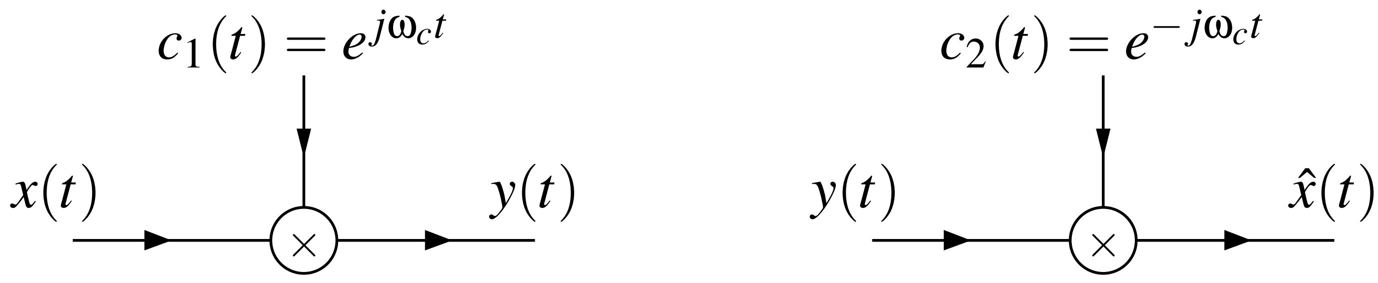
Section 6.8

Application: Amplitude Modulation (AM)

Motivation for Amplitude Modulation (AM)

- In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal.
- For example, voice/audio signals typically have information in the range of 0 to 22 kHz.
- Often, it is not practical to transmit such a signal using its original frequency range.
- Two potential problems with such an approach are:
  1. interference; and
  2. constraints on antenna length.
- Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference.
- Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals.
- For the preceding reasons, we often need to change the frequency range associated with a signal before transmission.

Trivial Amplitude Modulation (AM) System



Transmitter Receiver

- The transmitter is characterized by

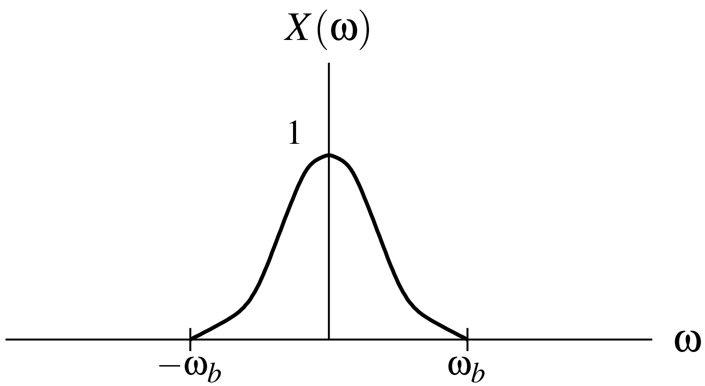
$$y(t) = e^{j\omega_c t}x(t) \quad \Longleftrightarrow \quad Y(\omega) = X(\omega - \omega_c).$$

- The receiver is characterized by

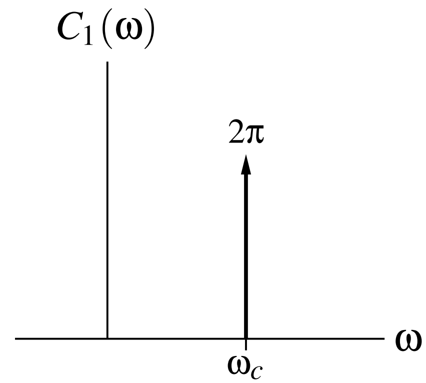
$$\hat{x}(t) = e^{-j\omega_c t}y(t) \quad \Longleftrightarrow \quad \hat{X}(\omega) = Y(\omega + \omega_c).$$

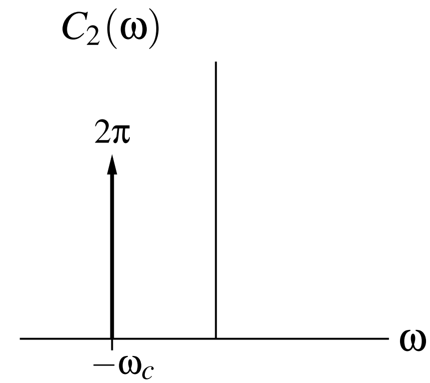
- Clearly, $\hat{x}(t) = e^{j\omega_c t}e^{-j\omega_c t}x(t) = x(t)$.

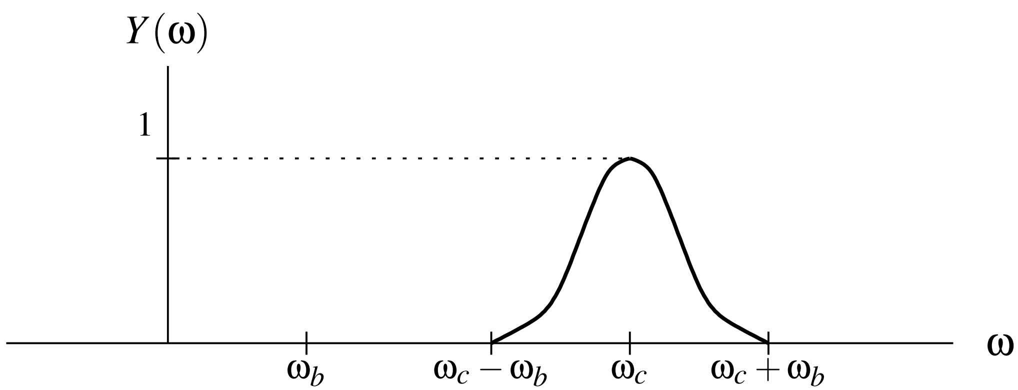
Trivial Amplitude Modulation (AM) System: Example



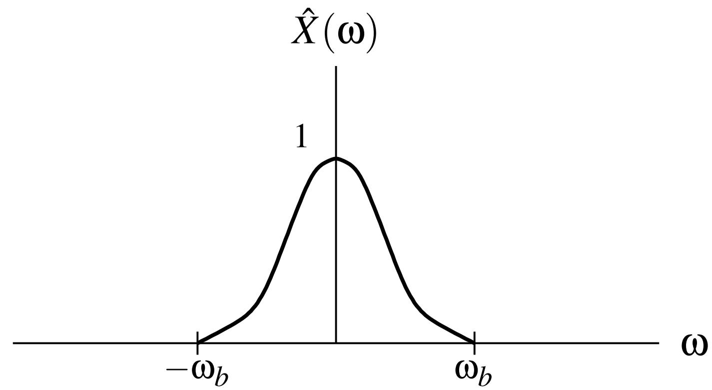
Transmitter Input





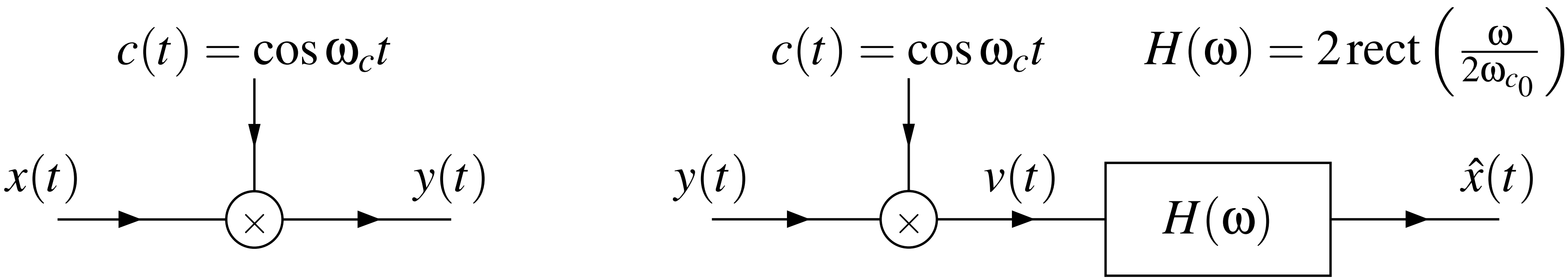


Transmitter Output



Receiver Output

Double-Sideband Suppressed-Carrier (DSB-SC) AM



Transmitter

Receiver

- Suppose that $X(\omega) = 0$ for all ω not in $[-\omega_b, \omega_b]$.
- The transmitter is characterized by

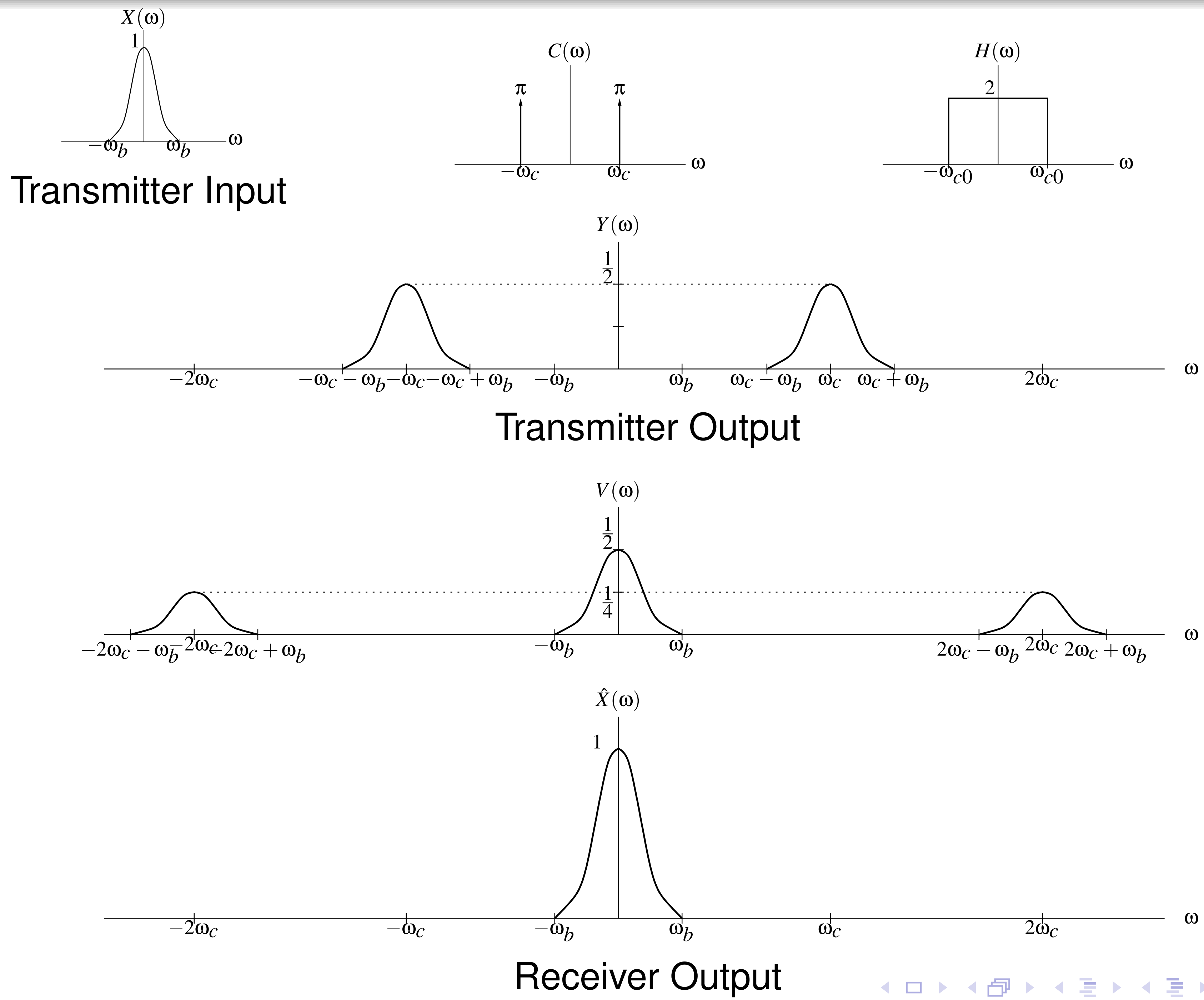
$$Y(\omega) = \tfrac{1}{2}\left[X(\omega+\omega_c) + X(\omega-\omega_c)\right].$$

- The receiver is characterized by

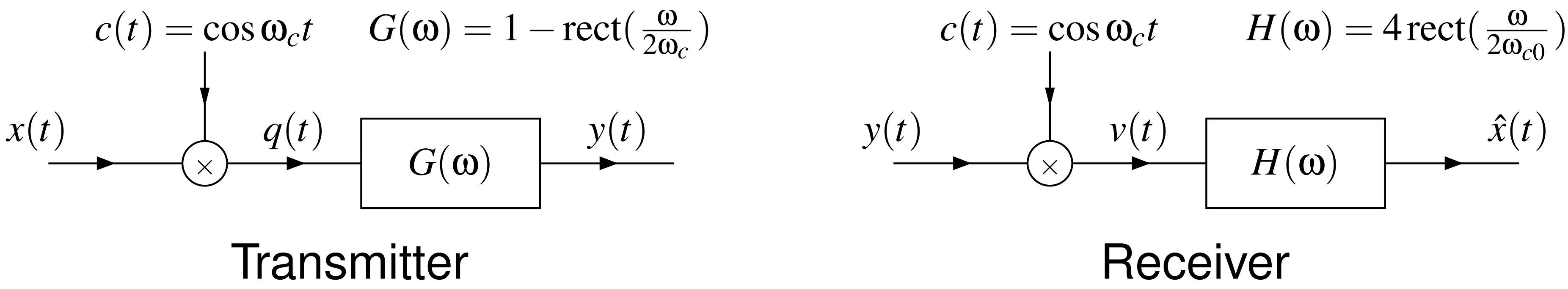
$$\hat{X}(\omega) = \left[Y(\omega+\omega_c) + Y(\omega-\omega_c)\right]\mathrm{rect}\left(\frac{\omega}{2\omega_{c_0}}\right).$$

- If $\omega_b < \omega_{c_0} < 2\omega_c - \omega_b$, we have $\hat{X}(\omega) = X(\omega)$.

DSB-SC AM: Example

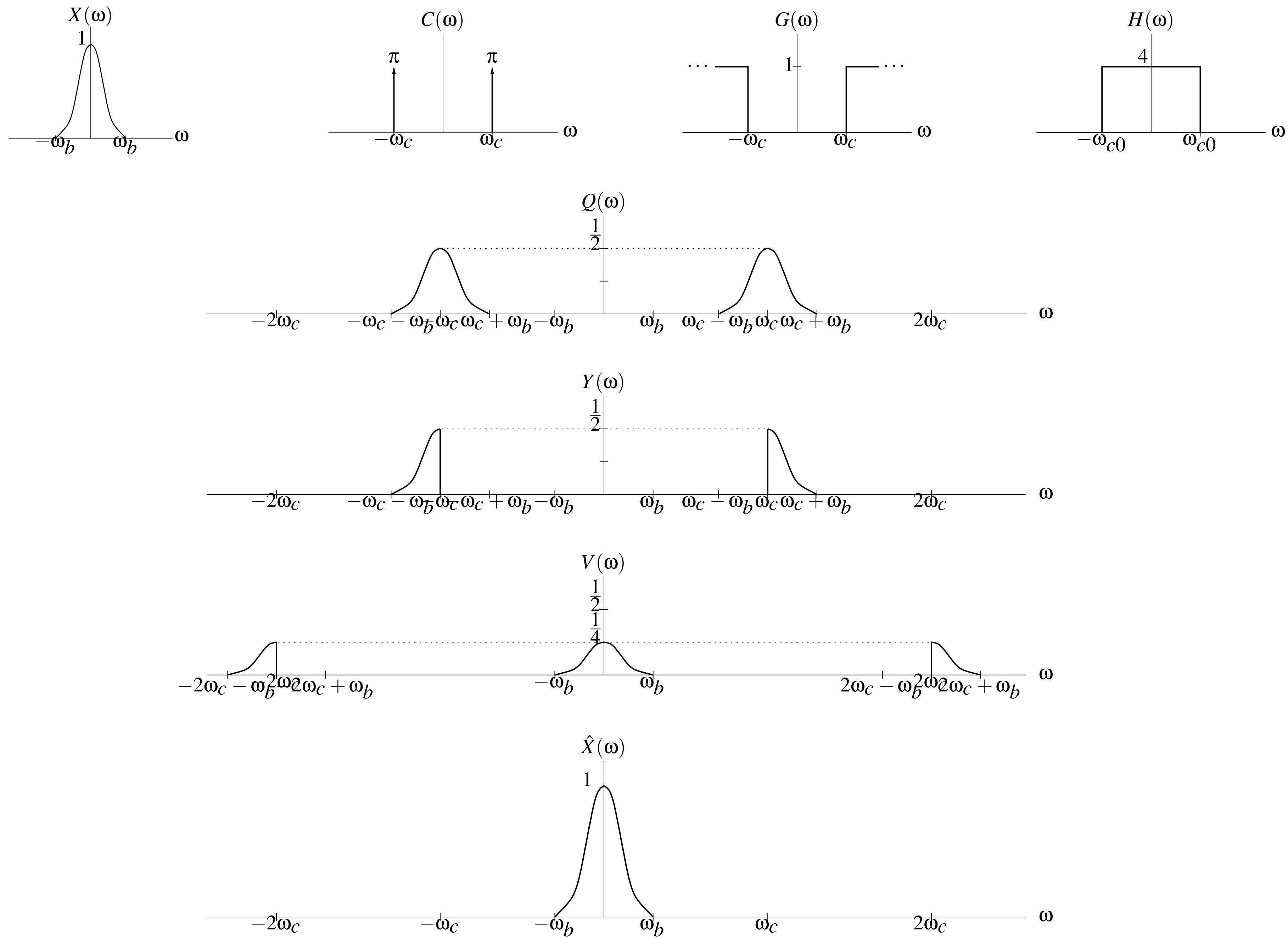


Single-Sideband Suppressed-Carrier (SSB-SC) AM



- The basic analysis the SSB-SC AM system is similar to the DSB-SC AM system.
- SSB-SC AM requires half as much bandwidth for the transmitted signal as DSB-SC AM.

SSB-SC AM: Example

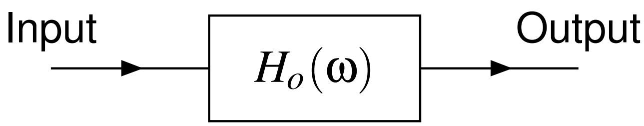


Section 6.9

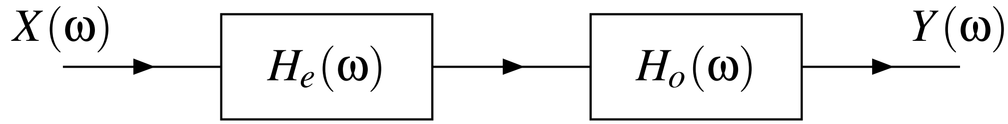
Application: Equalization

Equalization

- Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand.
- As a result, we would like to change the frequency response of the system to be something more desirable.
- This process of modifying the frequency response in this way is referred to as **equalization**. [Essentially, equalization is just a filtering operation.]
- Equalization is used in many applications.
- In real-world ***communication systems***, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel.
- In ***audio applications***, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, often we like to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.



Original System



New System with Equalization

- Let $H_{\text{o}}(\omega)$ denote the frequency response of ***original*** system (i.e., without equalization).
- Let $H_{\text{d}}(\omega)$ denote the ***desired*** frequency response.
- Let $H_{\text{e}}(\omega)$ denote the frequency response of the ***equalizer***.
- The new system with equalization has frequency response

$$H_{\text{new}}(\omega) = H_{\text{e}}(\omega)H_{\text{o}}(\omega).$$

- By choosing $H_{\text{e}}(\omega) = H_{\text{d}}(\omega)/H_{\text{o}}(\omega)$, the new system with equalization will have the frequency response

$$H_{\text{new}}(\omega) = [H_{\text{d}}(\omega)/H_{\text{o}}(\omega)]\,H_{\text{o}}(\omega) = H_{\text{d}}(\omega).$$

- In effect, by using an equalizer, we can obtain a new system with the frequency response that we desire.

Part 7

Partial Fraction Expansions (PFEs)

Motivation

- Sometimes it is beneficial to be able to express a rational function as a sum of ***lower-order*** rational functions.
- This can be accomplished using a type of decomposition known as a partial fraction expansion.
- Partial fraction expansions are often useful in the calculation of inverse Laplace and inverse Fourier transforms.

Strictly-Proper Rational Functions

- Consider a rational function

$$F(v) = \frac{\alpha_m v^m + \alpha_{m-1} v^{m-1} + \ldots + \alpha_1 v + \alpha_0}{\beta_n v^n + \beta_{n-1} v^{n-1} + \ldots + \beta_1 v + \beta_0}.$$

- The function $F(v)$ is said to be **strictly proper** if $m < n$ (i.e., the order of the numerator polynomial is strictly less than the order of the denominator polynomial).
- Through polynomial long division, any rational function can be written as the sum of a polynomial and a strictly-proper rational function.
- A ***strictly-proper*** rational function can be expressed as a sum of lower-order rational functions, with such an expression being called a partial fraction expansion.

- Any rational function can be expressed in the form of

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_0}{v^n + b_{m-1} v^{m-1} + \ldots + b_0}.$$

- Furthermore, the denominator polynomial $D(v) = v^n + b_{m-1} v^{m-1} + \ldots + b_0$ in the above expression for $F(v)$ can be factored to obtain

$$D(v) = (v - p_1)^{q_1} (v - p_2)^{q_2} \cdots (v - p_n)^{q_n},$$

where the p_k are distinct and the q_k are integers.

- If $F(v)$ has only simple poles, $q_1 = q_2 = \cdots = q_n = 1$.
- Suppose that $F(v)$ is strictly proper (i.e., $m < n$).
- In the determination of a partial fraction expansion of $F(v)$, there are ***two cases*** to consider:
  1. $F(v)$ has ***only simple poles***; and
  2. $F(v)$ has ***at least one repeated pole***.

- Suppose that $F(v)$ has only simple poles.
- Then, $D(v)$ is of the form

$$D(v) = (v-p_1)(v-p_2)\cdots(v-p_n),$$

where the p_k are distinct.
- In this case, $F(v)$ has a partial fraction expansion of the form

$$F(v) = \frac{A_1}{v-p_1} + \frac{A_2}{v-p_2} + \ldots + \frac{A_{n-1}}{v-p_{n-1}} + \frac{A_n}{v-p_n},$$

where

$$A_k = (v-p_k)F(v)\big|_{v=p_k}.$$

- Note that the (simple) pole p_k contributes a single term to the partial fraction expansion.

Repeated-Pole Case

- Suppose that $F(v)$ has at least one repeated pole.
- One can show that, in this case, $F(v)$ has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{11}}{v-p_1} + \frac{A_{12}}{(v-p_1)^2} + \ldots + \frac{A_{1q_1}}{(v-p_1)^{q_1}}\right] + \left[\frac{A_{21}}{v-p_2} + \ldots + \frac{A_{2q_2}}{(v-p_2)^{q_2}}\right] + \ldots + \left[\frac{A_{P1}}{v-p_P} + \ldots + \frac{A_{Pq_P}}{(v-p_P)^{q_P}}\right],$$

where

$$A_{kl} = \frac{1}{(q_k-l)!}\left[\frac{d^{q_k-l}}{dv^{q_k-l}}[(v-p_k)^{q_k}F(v)]\right]\Bigg|_{v=p_k}.$$

- Note that the q_kth-order pole p_k contributes q_k terms to the partial fraction expansion.
- Note that $n! = (n)(n-1)(n-2)\cdots(1)$ and $0! = 1$.

Part 8

Laplace Transform

Section 8.1

Introduction

- Another important mathematical tool in the study of signals and systems is known as the Laplace transform.
- The Laplace transform can be viewed as a ***generalization of the Fourier transform***.
- Due to its more general nature, the Laplace transform has a number of ***advantages*** over the Fourier transform.
- First, the Laplace transform representation exists for some signals that do not have Fourier transform representations. So, we can handle a ***larger class of signals*** with the Laplace transform.
- Second, since the Laplace transform is a more general tool, it can provide ***additional insights*** beyond those facilitated by the Fourier transform.

Motivation Behind Laplace Transform

- Earlier, we saw that complex exponentials are eigenfunctions of LTI systems.
- In particular, for a LTI system $\mathcal{H}$ with impulse response $h(t)$, we have that

$$e^{st} \xrightarrow{\mathcal{H}} H(s)e^{st} \quad \text{where} \quad H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

- Previously, we referred to $H(s)$ as the system function.
- As it turns out, $H(s)$ is the Laplace transform of $h(t)$.
- Since the Laplace transform has already appeared earlier in the context of LTI systems, it is clearly a useful tool.
- Furthermore, as we will see, the Laplace transform has many additional uses.

(Bilateral) Laplace Transform

- The (bilateral) **Laplace transform** of the function $x(t)$ is denoted as $\mathcal{L}\{x(t)\}$ or $X(s)$ and is defined as
$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$
- The **inverse Laplace transform** is then given by
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds,$$
where $\sigma = \operatorname{Re}\{s\}$. [Note: This is a ***contour integration***, since s is complex.]
- We refer to $x(t)$ and $X(s)$ as a **Laplace transform pair** and denote this relationship as
$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s).$$
- In practice, we do not usually compute the inverse Laplace transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

Bilateral and Unilateral Laplace Transforms

- Two different versions of the Laplace transform are commonly used:
  1. the ***bilateral*** (or ***two-sided***) Laplace transform; and
  2. the ***unilateral*** (or ***one-sided***) Laplace transform.
- The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions.
- As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the ***lower limit of integration***.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0.
- In this course, we will focus our attention primarily on the bilateral Laplace transform.
- We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations.
- Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean ***bilateral*** Laplace transform.

Relationship Between Laplace and Fourier Transforms

- Let $X(s)$ and $X_\mathsf{F}(\omega)$ denote the Laplace and Fourier transforms of $x(t)$, respectively.
- The function $X(s)$ evaluated at $s = j\omega$ (where ω is real) yields $X_\mathsf{F}(\omega)$. That is,
$$X(s)|_{s=j\omega} = X_\mathsf{F}(\omega).$$
- Due to the preceding relationship, the Fourier transform of $x(t)$ is sometimes written as $X(j\omega)$.
- The function $X(s)$ evaluated at an arbitrary complex value $s = \sigma + j\omega$ (where $\sigma = \mathrm{Re}\{s\}$ and $\omega = \mathrm{Im}\{s\}$) can also be expressed in terms of a Fourier transform involving $x(t)$. In particular, we have
$$X(s)|_{s=\sigma+j\omega} = X'_\mathsf{F}(\omega),$$
where $X'_\mathsf{F}(\omega)$ is the Fourier transform of $e^{-\sigma t}x(t)$.
- So, in general, the Laplace transform of $x(t)$ is the Fourier transform of an exponentially-weighted version of $x(t)$.
- Due to this weighting, the Laplace transform of a signal may exist when the Fourier transform of the same signal does not.

Section 8.2

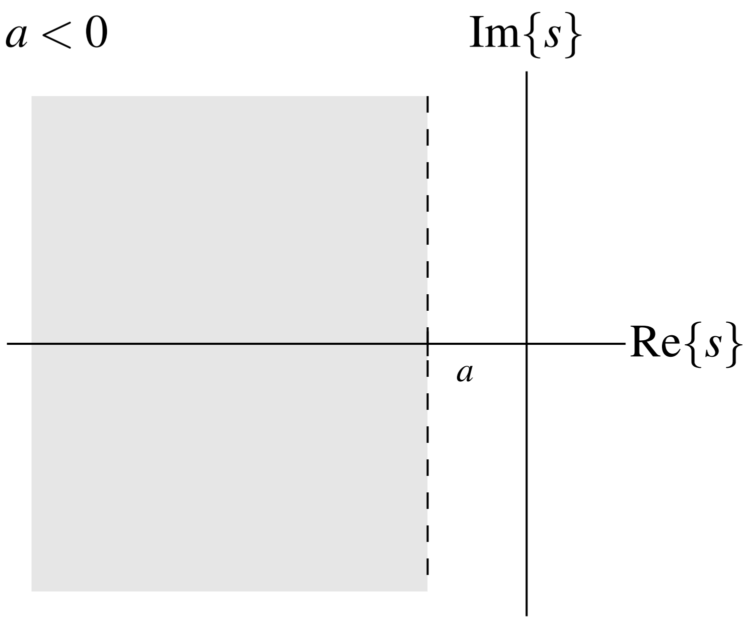
Region of Convergence (ROC)

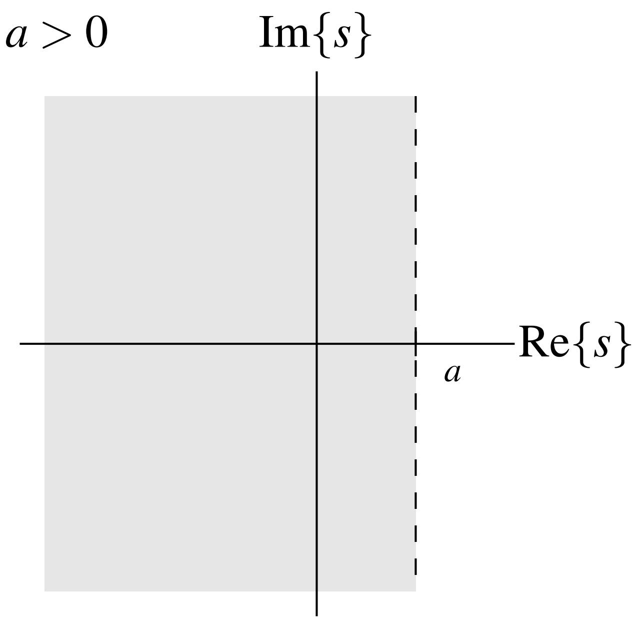
- The set R of all complex numbers s satisfying

$$\operatorname{Re}\{s\} < a$$

for some real constant a (or $a = \infty$) is said to be a **left-half plane (LHP)**.

- Some examples of LHPs are shown below.



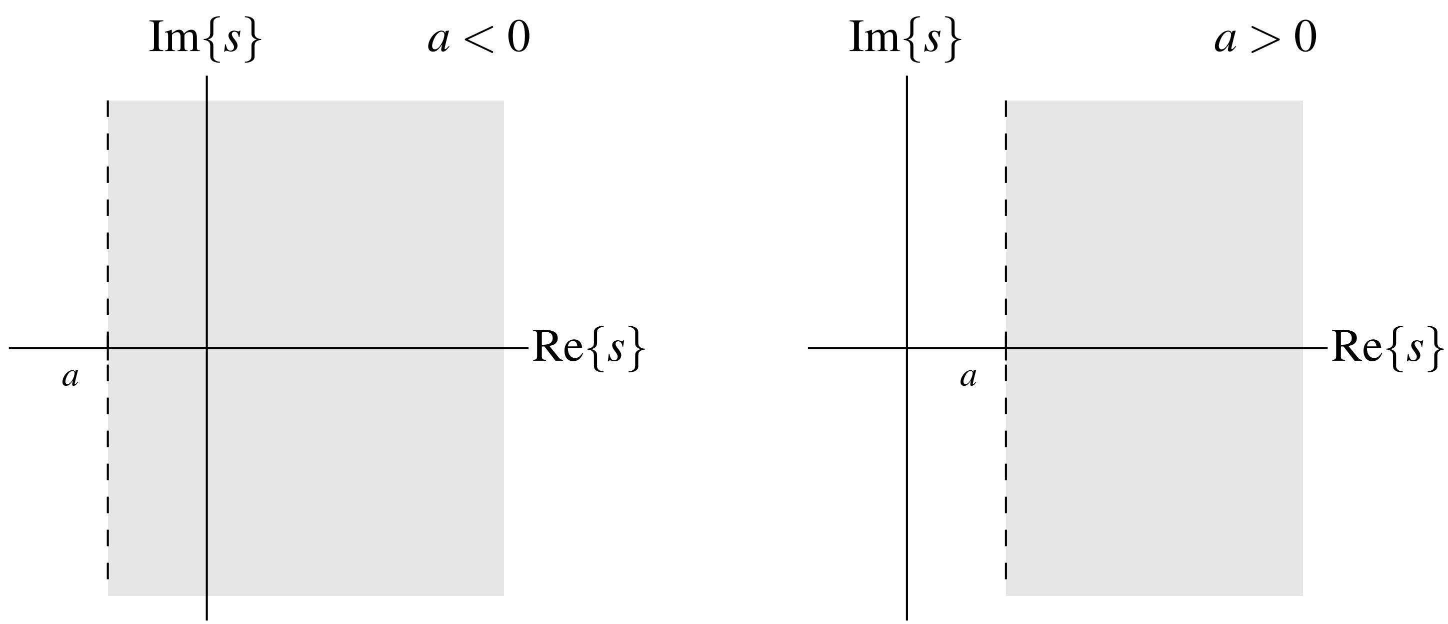


- The set R of all complex numbers s satisfying

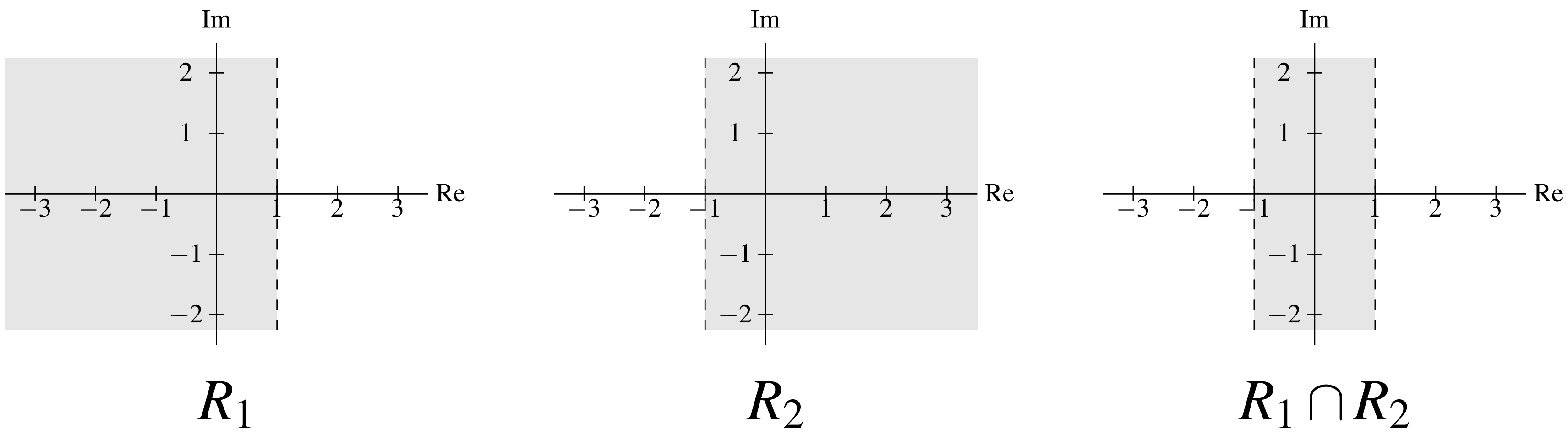
$$\operatorname{Re}\{s\} > a$$

for some real constant a (or $a = -\infty$) is said to be a **right-half plane (RHP)**.

- Some examples of RHPs are shown below.



- For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B.
- An illustrative example of set intersection is shown below.



Region of Convergence (ROC)

- As we saw earlier, for a signal $x(t)$, the complete specification of its Laplace transform $X(s)$ requires not only an algebraic expression for $X(s)$, but also the ROC associated with $X(s)$.
- Two very different signals can have the same algebraic expressions for $X(s)$.
- Now, we examine some of the constraints on the ROC (of the Laplace transform) for various classes of signals.

Properties of ROC

1. The ROC of the Laplace transform $X(s)$ consists of ***strips parallel to the imaginary axis*** in the complex plane.
2. If the Laplace transform $X(s)$ is a ***rational*** function, the ROC ***does not contain any poles***, and the ROC is ***bounded by poles or extends to infinity***.
3. If the signal $x(t)$ is ***finite duration*** and its Laplace transform $X(s)$ converges for some value of s, then $X(s)$ converges for ***all values*** of s (i.e., the ROC is the entire complex plane).
4. If the signal $x(t)$ is ***right sided*** and the (vertical) line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X(s) = \mathcal{L}\{x(t)\}$, then all values of s for which $\mathrm{Re}\{s\} > \sigma_0$ must also be in the ROC (i.e., the ROC is a ***right-half plane*** including $\mathrm{Re}\{s\} = \sigma_0$).
5. If the signal $x(t)$ is ***left sided*** and the (vertical) line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X(s) = \mathcal{L}\{x(t)\}$, then all values of s for which $\mathrm{Re}\{s\} < \sigma_0$ must also be in the ROC (i.e., the ROC is a ***left-half plane*** including $\mathrm{Re}\{s\} = \sigma_0$).

Properties of ROC (Continued)

6. If the signal $x(t)$ is ***two sided*** and the (vertical) line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X(s) = \mathcal{L}\{x(t)\}$, then the ROC will consist of a ***strip*** in the complex plane that includes the line $\mathrm{Re}\{s\} = \sigma_0$.
7. If the Laplace transform $X(s)$ of the signal $x(t)$ is ***rational***, then:
    1. If $x(t)$ is ***right sided***, the ROC of $X(s)$ is to the right of the rightmost pole of $X(s)$ (i.e., the ***right-half plane*** to the ***right of the rightmost pole***).
    2. If $x(t)$ is ***left sided***, the ROC of $X(s)$ is to the left of the leftmost pole of $X(s)$ (i.e., the ***left-half plane*** to the ***left of the leftmost pole***).

- Some of the preceding properties are ***redundant*** (e.g., properties 1, 2, 4, and 5 imply property 7).
- Since every function can be classified as one of finite duration, left sided, right sided, or two sided, we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of a LHP, RHP, vertical strip, the entire complex plane, or an empty set. Thus, the ROC must be a ***connected region***.

Section 8.3

Properties of Laplace Transform

Properties of Laplace Transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\mathrm{Re}\{s_0\}$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathrm{Re}\{s\}>0\}$

Property	
Initial Value Theorem	$x(0^+)=\lim\limits_{s\to\infty} sX(s)$
Final Value Theorem	$\lim\limits_{t\to\infty} x(t)=\lim\limits_{s\to 0} sX(s)$

Laplace Transform Pairs

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < -a$
8	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > -a$
9	$-t^n e^{-at}u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} < -a$
10	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
11	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
12	$[e^{-at}\cos\omega_0 t]u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$
13	$[e^{-at}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$

- If $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

  $$a_1x_1(t) + a_2x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a_1X_1(s) + a_2X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2,$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Laplace transform.
- The ROC always contains the intersection but could be larger (in the case that pole-zero cancellation occurs).

Time-Domain Shifting

- If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R, then

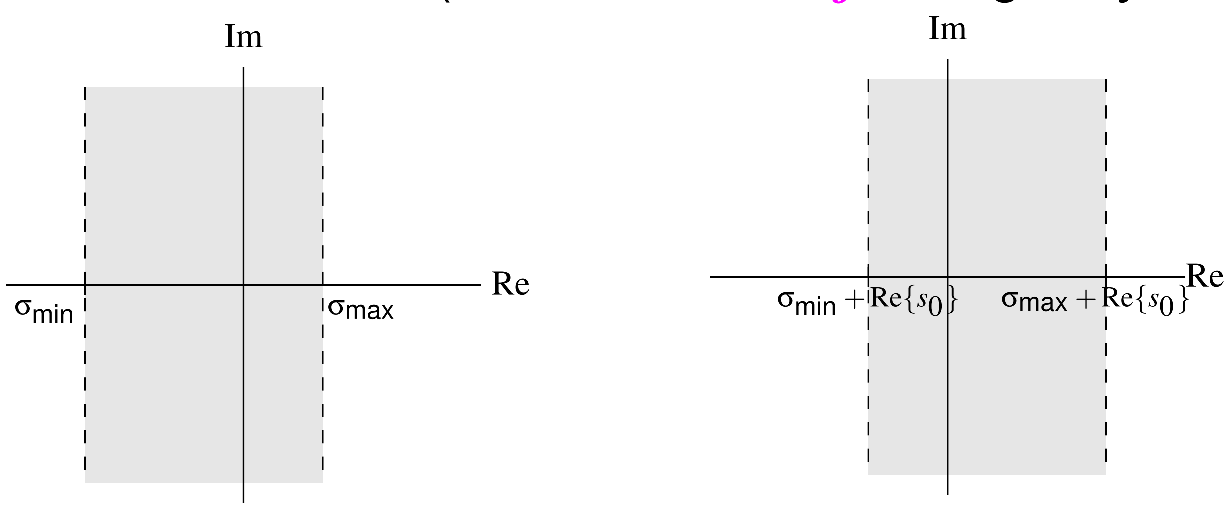
$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}X(s) \text{ with ROC } R,$$

  where t_0 is an arbitrary real constant.
- This is known as the **time-domain shifting property** of the Laplace transform.

- If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$e^{s_0 t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-s_0) \text{ with ROC } R+\operatorname{Re}\{s_0\},$$

  where s_0 is an arbitrary complex constant.
- This is known as the **Laplace-domain shifting property** of the Laplace transform.
- The ROCs are illustrated below. (The ROC is ***shifted*** right by $\operatorname{Re}\{s_0\}$.)

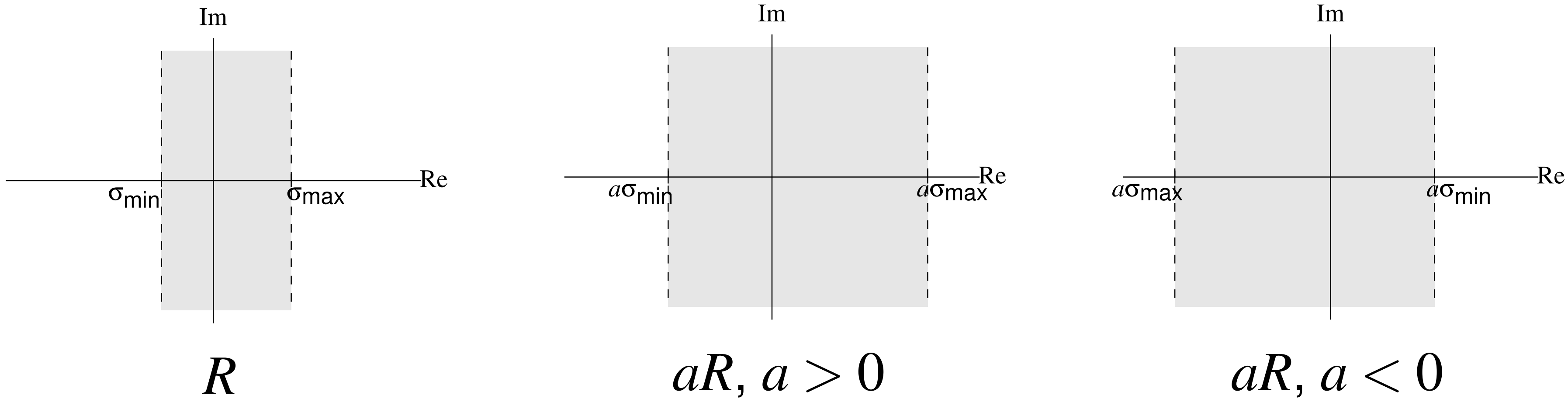


R $\qquad$ $R+\operatorname{Re}\{s_0\}$

- If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right) \quad \text{with ROC } R_1 = aR,$$

  where a is a nonzero real constant.
- This is known as the **(time-domain/Laplace-domain) scaling property** of the Laplace transform.
- The ROCs associated with scaling are illustrated below. (The ROC is ***scaled*** and ***possibly flips*** left to right.)



- If $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R, then

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*) \text{ with ROC } R.$$

- This is known as the **conjugation property** of the Laplace transform.

Time-Domain Convolution

- If $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

$$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s) \text{ with ROC containing } R_1 \cap R_2.$$

- This is known as the **time-domain convolution property** of the Laplace transform.
- The ROC always contains the intersection but can be larger than the intersection (if pole-zero cancellation occurs).
- Convolution in the time domain becomes ***multiplication*** in the Laplace domain. (This can make dealing with LTI systems much easier in the Laplace domain.)

- If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s) \text{ with ROC containing } R.$$

- This is known as the **time-domain differentiation property** of the Laplace transform.
- The ROC always contains R but can be larger than R (if pole-zero cancellation occurs).
- Differentiation in the time domain becomes multiplication by s in the Laplace domain. (This makes dealing with differential equations much easier in the Laplace domain.)

Laplace-Domain Differentiation

- If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds} \quad \text{with ROC } R.$$

- This is known as the **Laplace-domain differentiation property** of the Laplace transform.

Time-Domain Integration

- If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s) \text{ with ROC containing } R \cap \{\mathrm{Re}\{s\} > 0\}.$$

- This is known as the **time-domain integration property** of the Laplace transform.
- The ROC always contains at least $R \cap \{\mathrm{Re}\{s\} > 0\}$ but can be larger (if pole-zero cancellation occurs).
- Integration in the time domain becomes division by s in the Laplace domain. (This makes dealing with integral equations much easier to handle in the Laplace domain.)

- If $x(t) = 0$ ***for all*** $t < 0$ and $x(t)$ contains ***no impulses or higher order singularities at the origin***, then

$$x(0^+) = \lim_{s \to \infty} sX(s),$$

where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t.

- This result is known as the **initial value theorem**.

Final Value Theorem

- If $x(t) = 0$ ***for all*** $t < 0$ and $x(t)$ has a ***finite limit*** as $t \to \infty$, then
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$
- This result is known as the **final value theorem**.
- Sometimes the initial and final value theorems are useful for checking for errors in Laplace transform calculations. For example, if we had made a mistake in computing $X(s)$, the values obtained from the initial and final value theorems would most likely disagree with the values obtained directly from the original expression for $x(t)$.

More Laplace Transform Examples

Section 8.4

Determination of Inverse Laplace Transform

Finding Inverse Laplace Transform

- Recall that the inverse Laplace transform $x(t)$ of $X(s)$ is given by

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds,$$

  where $\sigma = \mathrm{Re}\{s\}$.
- Unfortunately, the above contour integration can often be ***quite tedious*** to compute.
- Consequently, we do not usually compute the inverse Laplace transform directly using the above equation.
- For rational functions, the inverse Laplace transform can be more easily computed using ***partial fraction expansions***.
- Using a partial fraction expansion, we can express a rational function as a sum of lower-order rational functions whose inverse Laplace transforms can typically be found in a table.

Section 8.5

Laplace Transform and LTI Systems

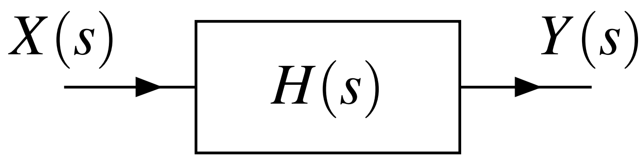
System Function of LTI Systems

- Consider a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Let $X(s)$, $Y(s)$, and $H(s)$ denote the Laplace transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.
- Since $y(t) = x(t) * h(t)$, the system is characterized in the Laplace domain by

$$Y(s) = X(s)H(s).$$

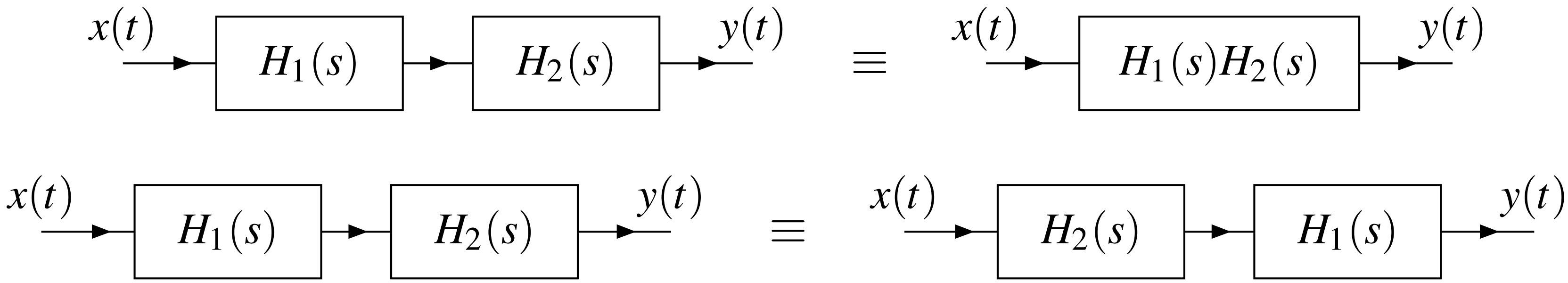
- As a matter of terminology, we refer to $H(s)$ as the **system function** (or **transfer function**) of the system (i.e., the system function is the Laplace transform of the impulse response).
- When viewed in the Laplace domain, a LTI system forms its output by multiplying its input with its system function.
- A LTI system is completely characterized by its system function $H(s)$.
- If the ROC of $H(s)$ includes the imaginary axis, then $H(s)|_{s=j\omega}$ is the ***frequency response*** of the LTI system.

- Consider a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$.
- Let $X(s)$, $Y(s)$, and $H(s)$ denote the Laplace transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.
- Often, it is convenient to represent such a system in block diagram form in the Laplace domain as shown below.

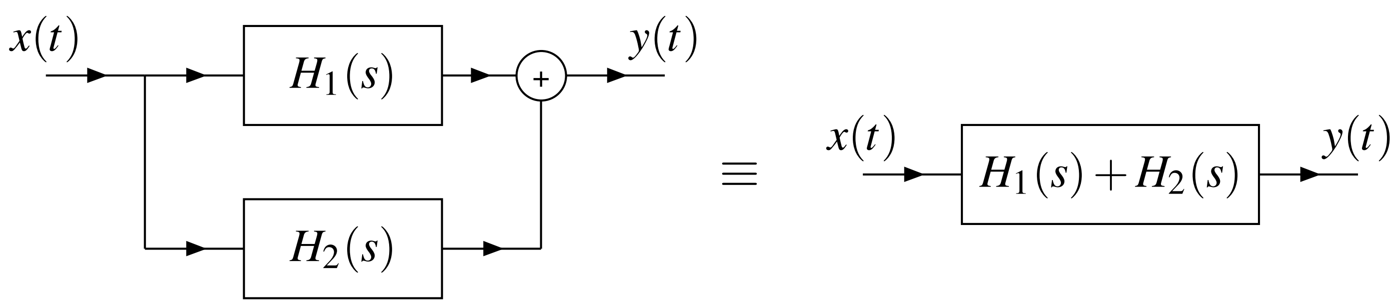


- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

- The ***series*** interconnection of the LTI systems with system functions $H_1(s)$ and $H_2(s)$ is the LTI system with system function $H(s) = H_1(s)H_2(s)$. That is, we have the equivalences shown below.



- The ***parallel*** interconnection of the LTI systems with impulse responses $H_1(s)$ and $H_2(s)$ is a LTI system with the system function $H(s) = H_1(s) + H_2(s)$. That is, we have the equivalence shown below.



Causality

- If a LTI system is ***causal***, its impulse response is causal, and therefore ***right sided***. From this, we have the result below.
- **Theorem.** The ROC associated with the system function of a ***causal*** system is a ***right-half plane***.
- In general, the ***converse*** of the above theorem is ***not necessarily true***. That is, it is not always true that a right-half plane ROC is associated with a causal system function.
- If $H(s)$ is ***rational***, however, we have that the converse does hold, as indicated by the theorem below.
- **Theorem.** For a system with a ***rational*** system function $H(s)$, ***causality*** of the system is ***equivalent*** to the ROC of $H(s)$ being the ***right-half plane*** to the right of the rightmost pole.

BIBO Stability

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is ***BIBO stable*** if and only if the ROC of its system function $H(s)$ includes the entire ***imaginary axis*** (i.e., $\mathrm{Re}\{s\} = 0$).
- **Theorem.** A ***causal*** system with a ***rational*** system function $H(s)$ is BIBO stable if and only if all of the poles of $H(s)$ lie in the left half of the plane (i.e., all of the poles have ***negative real parts***).

Invertibility

- Consider a LTI system $\mathcal{H}$ with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Let $X(s)$, $Y(s)$, and $H(s)$ denote the Laplace transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.
- The system is invertible if there exists another LTI system with system function $H^{\text{inv}}(s)$ such that
$$H(s)H^{\text{inv}}(s) = 1.$$
- Thus, the system function $H^{\text{inv}}(s)$ of $\mathcal{H}^{-1}$ is given by
$$H^{\text{inv}}(s) = \frac{1}{H(s)}.$$
- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is ***not necessarily unique***.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in ***one specific choice*** of inverse system (due to these additional constraints of stability and/or causality).

System Function and Differential Equation Representations of LTI Systems

- Many LTI systems of practical interest can be represented using an *N***th-order linear differential equation with constant coefficients**.
- Consider a system with input $x(t)$ and output $y(t)$ that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t) \quad \text{where} \quad M \leq N.$$

- Let $h(t)$ denote the impulse response of the system, and let $X(s)$, $Y(s)$, and $H(s)$ denote the Laplace transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.
- One can show that $H(s)$ is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

- Observe that, for a system of the form considered above, the system function is always ***rational***.

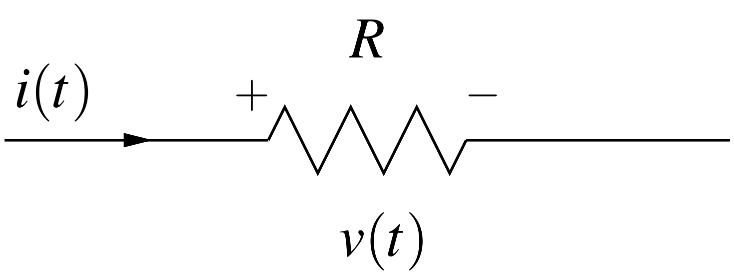
Section 8.6

Application: Circuit Analysis

- A **resistor** is a circuit element that opposes the flow of electric current.
- A resistor is governed by the relationship

$$v(t) = Ri(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad V(s) = RI(s)$$

where R, $v(t)$ and $i(t)$ denote the resistance of, voltage across, and current through the resistor, respectively.
- In circuit diagrams, a resistor is denoted by the symbol shown below.



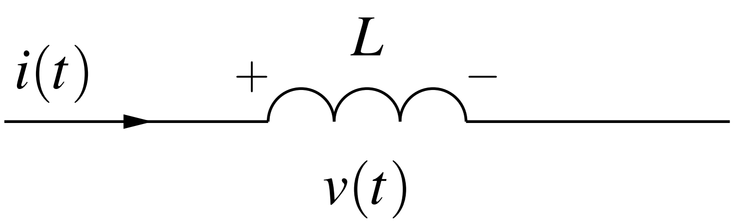
- An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa.
- The inductor is governed by the relationship

$$v(t) = L\tfrac{d}{dt}i(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad V(s) = sLI(s)$$

or equivalently

$$i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad I(s) = \tfrac{1}{sL}V(s)$$

where L, $v(t)$, and $i(t)$ denote the inductance of, voltage across, and current through the inductor, respectively.
- In circuit diagrams, a inductor is denoted by the symbol shown below.



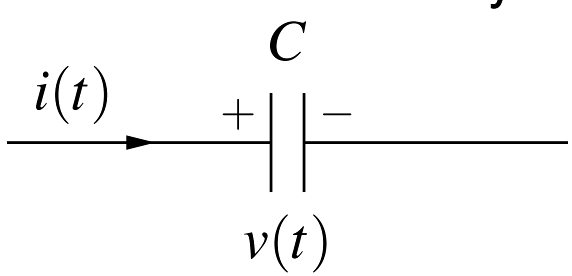
- A **capacitor** is a circuit element that stores electric charge.
- The capacitor is governed by the relationship

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad V(s) = \frac{1}{sC}I(s)$$

  or equivalently

$$i(t) = C\frac{d}{dt}v(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad I(s) = sCV(s)$$

  where C, $v(t)$, and $i(t)$ denote the capacitance of, voltage across, and current through the capacitor, respectively.
- In circuit diagrams, a capacitor is denoted by the symbol shown below.

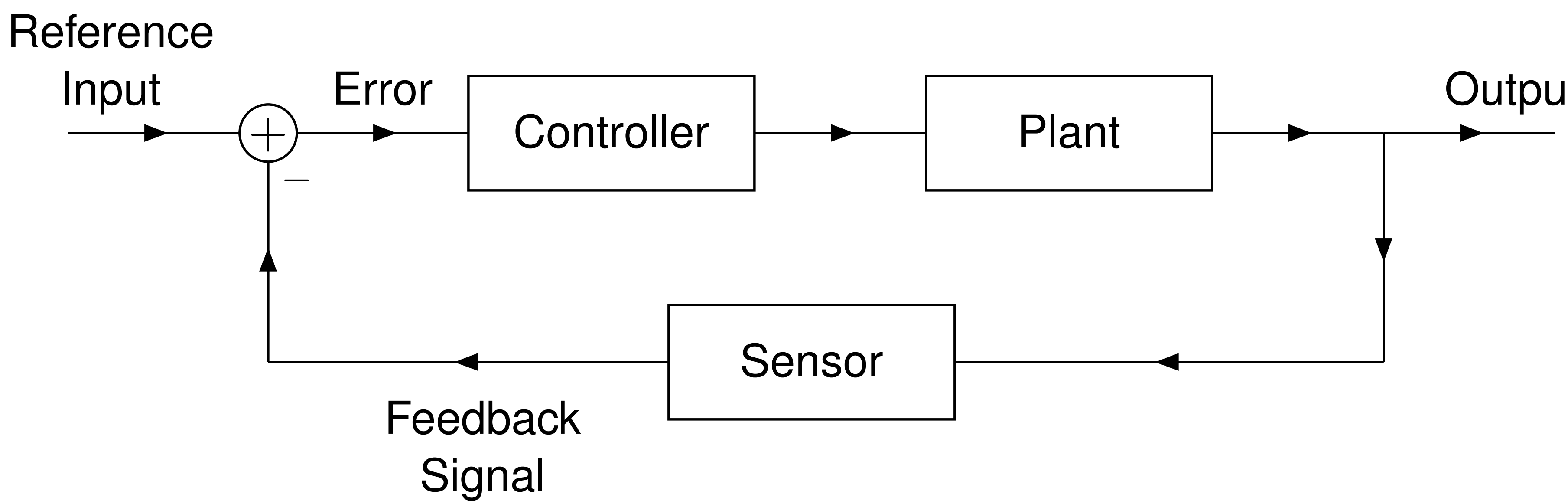


- The Laplace transform is very useful tool for circuit analysis.
- The utility of the Laplace transform is partly due to the fact that the ***differential/integral*** equations that describe inductors and capacitors are much simpler to express in the Laplace domain than in the time domain.

Section 8.7

Application: Analysis of Control Systems

Feedback Control Systems



- **input**: ***desired value*** of the quantity to be controlled
- **output**: ***actual value*** of the quantity to be controlled
- **error**: ***difference*** between the desired and actual values
- **plant**: system to be controlled
- **sensor**: device used to measure the actual output
- **controller**: device that monitors the error and changes the input of the plant with the goal of forcing the error to zero

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the Laplace domain than in the time domain.
- Therefore, the Laplace domain is extremely useful for the stability analysis of systems.

Section 8.8

Unilateral Laplace Transform

Unilateral Laplace Transform

- The **unilateral Laplace transform** of the signal $x(t)$, denoted $\mathcal{UL}\{x(t)\}$ or $X(s)$, is defined as

$$\mathcal{UL}\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt.$$

- The unilateral Laplace transform is related to the bilateral Laplace transform as follows:

$$\mathcal{UL}\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt = \mathcal{L}\{x(t)u(t)\}.$$

- In other words, the unilateral Laplace transform of the signal $x(t)$ is simply the bilateral Laplace transform of the signal $x(t)u(t)$.
- Since $\mathcal{UL}\{x(t)\} = \mathcal{L}\{x(t)u(t)\}$ and $x(t)u(t)$ is always a ***right-sided*** signal, the ROC associated with $\mathcal{UL}\{x(t)\}$ is always a ***right-half plane***.
- For this reason, we often ***do not explicitly indicate the ROC*** when working with the unilateral Laplace transform.

- With the unilateral Laplace transform, the same inverse transform equation is used as in the bilateral case.
- The unilateral Laplace transform is ***only invertible for causal signals***. In particular, we have

$$\begin{aligned}
\mathcal{UL}^{-1}\{\mathcal{UL}\{x(t)\}\} &= \mathcal{UL}^{-1}\{\mathcal{L}\{x(t)u(t)\}\} = \mathcal{L}^{-1}\{\mathcal{L}\{x(t)u(t)\}\} \\
&= x(t)u(t) \\
&= \begin{cases} x(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}
\end{aligned}$$

- For a noncausal signal $x(t)$, we can only recover $x(t)$ for $t > 0$.

Unilateral Laplace Transform (Continued 2)

- Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties.
- Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways.

Unilateral Laplace Transform Properties

Property	Time Domain	Laplace Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$
Time/Frequency-Domain Scaling	$x(at)$, $a>0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1(t)*x_2(t)$, $x_1(t)$ and $x_2(t)$ are causal	$X_1(s)X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$
Time-Domain Integration	$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Unilateral Laplace Transform Pairs

Pair	$x(t),\ t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos \omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin \omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at} \cos \omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

Solving Differential Equations Using the Unilateral Laplace Transform

- Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations.
- One common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions.

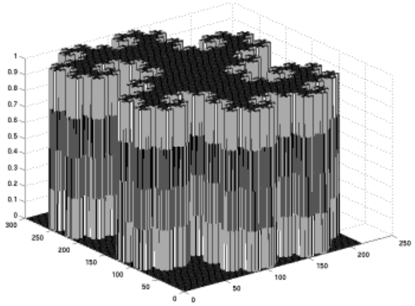
Part 9

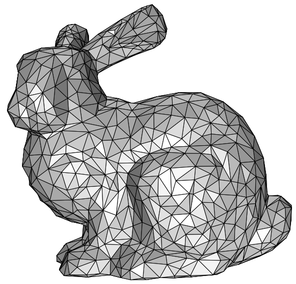
And Life Goes On...

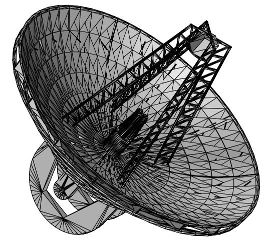
- If you did not suffer permanent emotional scarring as a result of using these lecture slides and you happen to be a student at the University of Victoria, you might consider taking the following course (developed by the author of these lecture slides) as one of your technical electives (in third or fourth year):

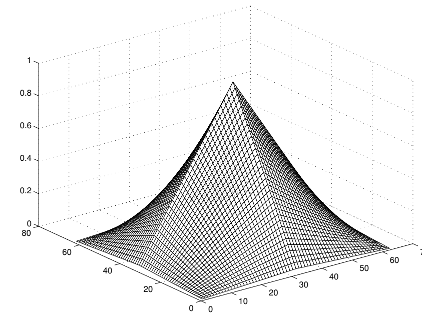
  ELEC 486: Multiresolution Signal and Geometry Processing with Software Applications (in C++)

- Some further information about ELEC 486 can be found *on the next slide*, including the URL of the course web page.









ELEC 486/586: Multiresolution Signal and Geometry Processing with Software Applications (in C++)

- normally offered in Summer (May-August) term; only prerequisite ELEC 310
- subdivision surfaces and subdivision wavelets
  - 3D computer graphics, animation, gaming (Toy Story, Blender software)
  - geometric modelling, visualization, computer-aided design
- multirate signal processing and wavelet systems
  - sampling rate conversion (audio processing, video transcoding)
  - signal compression (JPEG 2000, FBI fingerprint compression)
  - communication systems (transmultiplexers for CDMA, FDMA, TDMA)
- C++ (classes, templates, standard library), OpenGL, GLUT, CGAL
- software applications (using C++)
- for more information, visit course web page:
  `http://www.ece.uvic.ca/~mdadams/courses/wavelets`