Lecture Slides for Signals and Systems

Edition 2.0

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Published by the University of Victoria, Victoria, British Columbia, Canada

This document was typeset with LaTeX.

ISBN 978-1-55058-661-9 (paperback)
ISBN 978-1-55058-662-6 (PDF)

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1. M. D. Adams, *Lecture Slides for Programming in C++ (Version 2019-02-04)*, University of Victoria, Victoria, BC, Canada, Feb. 2019, xxii + 2516 slides, ISBN 978-1-55058-640-4 (print), ISBN 978-1-55058-641-1 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/cppbook.
2. M. D. Adams, *Multiresolution Signal and Geometry Processing: Filter Banks, Wavelets, and Subdivision (Version 2013-09-26)*, University of Victoria, Victoria, BC, Canada, Sept. 2013, xxxviii + 538 pages, ISBN 978-1-55058-507-0 (print), ISBN 978-1-55058-508-7 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/waveletbook.

Other Textbooks and Lecture Slides by the Author II

3. M. D. Adams, *Lecture Slides for Multiresolution Signal and Geometry Processing (Version 2015-02-03)*, University of Victoria, Victoria, BC, Canada, Feb. 2015, xi + 587 slides, ISBN 978-1-55058-535-3 (print), ISBN 978-1-55058-536-0 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/waveletbook.
4. M. D. Adams, *Continuous-Time Signals and Systems*, Edition 2.0, University of Victoria, Victoria, BC, Canada, Feb. 2020, xxx + 366 pages, ISBN 978-1-55058-657-2 (print), ISBN 978-1-55058-658-9 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/sigsysbook.

Part 0

Preface

About These Lecture Slides

- This document constitutes a detailed set of lecture slides on signals and systems, covering both the continuous-time and discrete-time cases.
- These slides are organized in such a way as to facilitate the teaching of a course that covers: only the continuous-time case, or only the discrete-time case, or both the continuous-time and discrete-time cases.
- To teach a course on only the continuous-time case, these slides can be used in conjunction with the following textbook:
  - M. D. Adams, *Continuous-Time Signals and Systems*, Edition 2.0, University of Victoria, Victoria, BC, Canada, Feb. 2020. xxx + 366 pages. ISBN 978-1-55058-657-2 (print), ISBN 978-1-55058-658-9 (PDF). Available online from `http://www.ece.uvic.ca/~mdadams/sigsysbook`.
- The author is currently in the process of developing a new textbook that covers both the continuous-time and discrete-time cases. These lecture slides are also intended for use with this new textbook, when it becomes available.

Typesetting Conventions

- In a definition, the term being defined is often typeset in a font **like this**.
- To emphasize particular words, the words are typeset in a font ***like this***.

Part 1

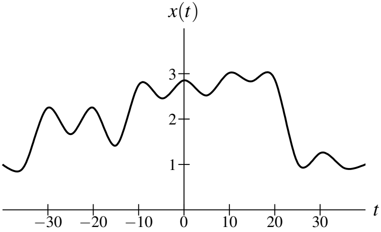
Introduction

Signals

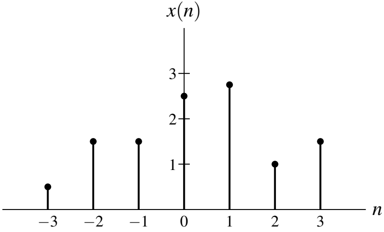
- A **signal** is a function of one or more variables that conveys information about some (usually physical) phenomenon.
- For a function f, in the expression $f(t_1, t_2, \ldots, t_n)$, each of the $\{t_k\}$ is called an **independent variable**, while the function value itself is referred to as a **dependent variable**.
- Some examples of signals include:
  - a voltage or current in an electronic circuit
  - the position, velocity, or acceleration of an object
  - a force or torque in a mechanical system
  - a flow rate of a liquid or gas in a chemical process
  - a digital image, digital video, or digital audio
  - a stock market index

Classification of Signals

- Number of independent variables (i.e., dimensionality):
  - A signal with *one* independent variable is said to be **one dimensional** (e.g., audio).
  - A signal with *more than one* independent variable is said to be **multi-dimensional** (e.g., image).
- Continuous or discrete independent variables:
  - A signal with *continuous* independent variables is said to be **continuous time (CT)** (e.g., voltage waveform).
  - A signal with *discrete* independent variables is said to be **discrete time (DT)** (e.g., stock market index).
- Continuous or discrete dependent variable:
  - A signal with a *continuous* dependent variable is said to be **continuous valued** (e.g., voltage waveform).
  - A signal with a *discrete* dependent variable is said to be **discrete valued** (e.g., digital image).
- A *continuous-valued CT* signal is said to be **analog** (e.g., voltage waveform).
- A *discrete-valued DT* signal is said to be **digital** (e.g., digital audio).

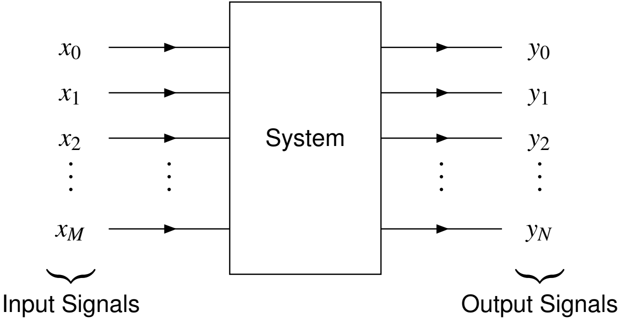


Continuous-Time (CT) Signal



Discrete-Time (DT) Signal

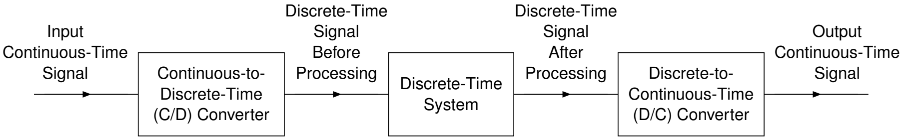
- A **system** is an entity that processes one or more input signals in order to produce one or more output signals.



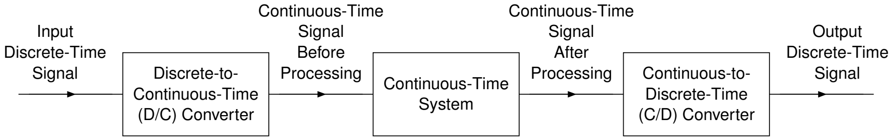
Classification of Systems

- Number of inputs:
  - A system with ***one*** input is said to be **single input (SI)**.
  - A system with ***more than one*** input is said to be **multiple input (MI)**.
- Number of outputs:
  - A system with ***one*** output is said to be **single output (SO)**.
  - A system with ***more than one*** output is said to be **multiple output (MO)**.
- Types of signals processed:
  - A system can be classified in terms of the ***types of signals*** that it processes.
  - Consequently, terms such as the following (which describe signals) can also be used to describe systems:
    - one-dimensional and multi-dimensional,
    - continuous-time (CT) and discrete-time (DT), and
    - analog and digital.
  - For example, a continuous-time (CT) system processes CT signals and a discrete-time (DT) system processes DT signals.

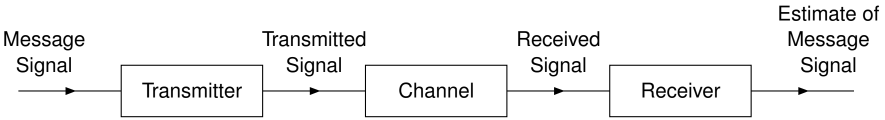
Signal Processing Systems



Processing a Continuous-Time Signal With a Discrete-Time System

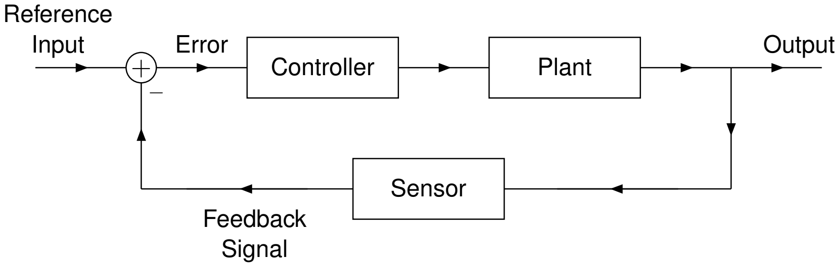


Processing a Discrete-Time Signal With a Continuous-Time System



General Structure of a Communication System

Control Systems



General Structure of a Feedback Control System

Why Study Signals and Systems?

- Engineers build systems that process/manipulate signals.
- We need a formal mathematical framework for the study of such systems.
- Such a framework is necessary in order to ensure that a system will meet the required specifications (e.g., performance and safety).
- If a system fails to meet the required specifications or fails to work altogether, negative consequences usually ensue.
- When a system fails to operate as expected, the consequences can sometimes be catastrophic.

System Failure Example: Tacoma Narrows Bridge

- The (original) Tacoma Narrows Bridge was a suspension bridge linking Tacoma and Gig Harbor (WA, USA).
- This mile-long bridge, with a 2,800-foot main span, was the third largest suspension bridge at the time of opening.
- Construction began in Nov. 1938 and took about 19 months to build at a cost of $6,400,000.
- On July 1, 1940, the bridge opened to traffic.
- On Nov. 7, 1940 at approximately 11:00, the bridge collapsed during a moderate (42 miles/hour) wind storm.
- The bridge was supposed to withstand winds of up to 120 miles/hour.
- The collapse was due to wind-induced vibrations and an ***unstable mechanical system***.
- Repair of the bridge was not possible.
- Fortunately, a dog trapped in an abandoned car was the only fatality.

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Part 2

Preliminaries

Section 2.1

Signals

- Earlier, we were introduced to CT and DT signals.
- A CT signal is called a **function**.
- A DT signal is called a **sequence**.
- Although, strictly speaking, a sequence is a special case of a function (where the domain of the function is the integers), we will use the term function exclusively to mean a function that is not a sequence.
- The nth element of a sequence x is denoted as either $x(n)$ or x_n.

Remarks on Notation for Functions and Sequences

- For a real-valued function f of a real variable and an arbitrary real number t, the expression f denotes the function f itself and the expression $f(t)$ denotes the value of the function f evaluated at the point t.
- That is, f is a ***function*** and $f(t)$ is a ***number***.
- Unfortunately, the practice of using $f(t)$ to denote the function f is quite common, although strictly speaking this is an abuse of notation.
- In contexts where imprecise notation may lead to problems, one should be careful to clearly distinguish between a function and its value.
- For the real-valued functions f and g of a real variable and an arbitrary real number t:
  - The expression $f+g$ denotes a ***function***, namely, the function formed by adding the functions f and g.
  - The expression $f(t)+g(t)$ denotes a ***number***, namely, the sum of: 1) the value of the function f evaluated at t; and 2) the value of the function g evaluated at t.
- Similar comments as the ones made above for functions also hold in the case of sequences.

- To express that two functions f and g are equal, we can write either:
  1. $f = g$; or
  2. $f(t) = g(t)$ for all t.
- Of the preceding two expressions, the first (i.e., $f = g$) is usually preferable, as it is less verbose.
- For the functions f and g and an operation $\circ$ that is defined pointwise for functions (such as addition, subtraction, multiplication, and division), the following relationship holds:

$$(f \circ g)(t) = f(t) \circ g(t).$$

- Some operations $\circ$ involving functions (such as convolution, to be discussed later) cannot be defined in a pointwise manner, in which case $(f \circ g)(t)$ is a valid mathematical expression, while $f(t) \circ g(t)$ is not.
- Again, similar comments as the ones made above for functions also hold in the case of sequences.

Section 2.2

Properties of Signals

Even Symmetry

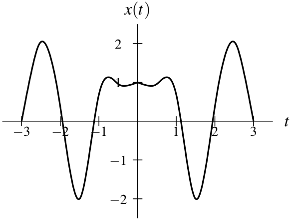
- A function x is said to be **even** if it satisfies

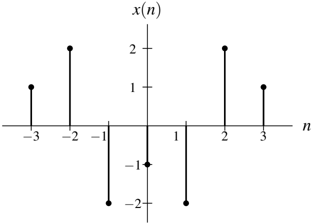
$$x(t) = x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- Geometrically, the graph of an even signal is ***symmetric*** about the origin.
- Some examples of even signals are shown below.





Odd Symmetry

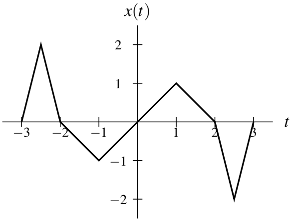
- A function x is said to be **odd** if it satisfies

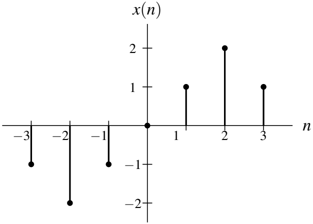
$$x(t) = -x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- Geometrically, the graph of an odd signal is ***antisymmetric*** about the origin.
- An odd signal x must be such that $x(0) = 0$.
- Some examples of odd signals are shown below.





Conjugate Symmetry

- A function x is said to be **conjugate symmetric** if it satisfies

$$x(t) = x^*(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **conjugate symmetric** if it satisfies

$$x(n) = x^*(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- The real part of a conjugate symmetric function or sequence is even.
- The imaginary part of a conjugate symmetric function or sequence is odd.
- An example of a conjugate symmetric function is a complex sinusoid $x(t) = \cos \omega t + j \sin \omega t$, where ω is a real constant.

- A function x is said to be **periodic** with **period** T (or T**-periodic**) if, for some strictly-positive real constant T, the following condition holds:

$$x(t) = x(t+T) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

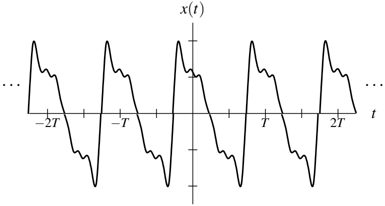
- A T-periodic function x is said to have **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$.
- A sequence x is said to be **periodic** with **period** N (or N**-periodic**) if, for some strictly-positive integer constant N, the following condition holds:

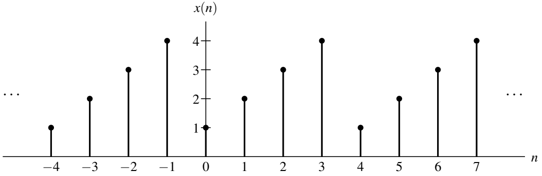
$$x(n) = x(n+N) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$.
- A function/sequence that is not periodic is said to be **aperiodic**.

Periodic Signals (Continued 1)

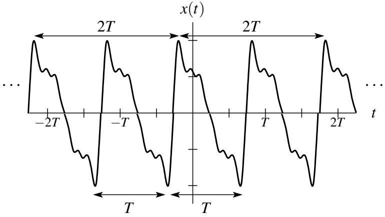
- Some examples of periodic signals are shown below.





Periodic Signals (Continued 2)

- The period of a periodic signal is ***not unique***. That is, a signal that is periodic with period T is also periodic with period kT, for every (strictly) positive integer k.



- The smallest period with which a signal is periodic is called the **fundamental period** and its corresponding frequency is called the **fundamental frequency**.

Part 3

Continuous-Time (CT) Signals and Systems

Section 3.1

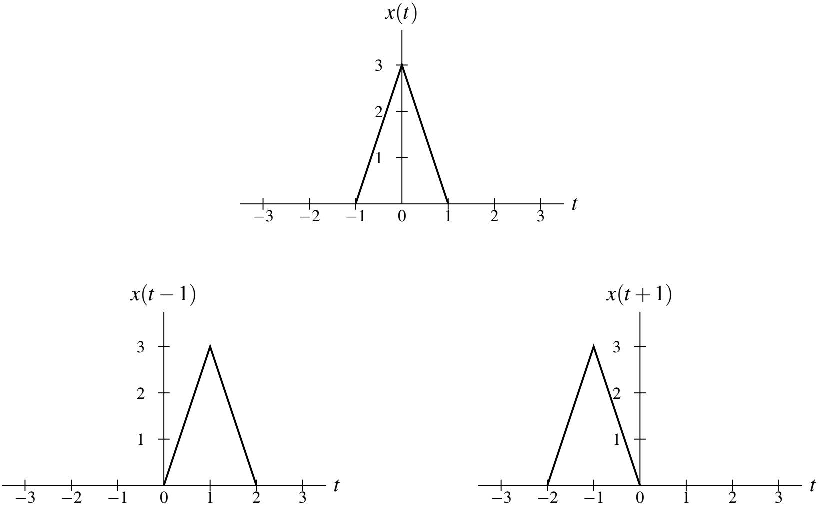
Independent- and Dependent-Variable Transformations

- **Time shifting** (also called **translation**) maps the input function x to the output function y as given by

$$y(t) = x(t - b),$$

  where b is a real number.
- Such a transformation shifts the function (to the left or right) along the time axis.
- If $b > 0$, y is ***shifted to the right*** by $|b|$, relative to x (i.e., delayed in time).
- If $b < 0$, y is ***shifted to the left*** by $|b|$, relative to x (i.e., advanced in time).

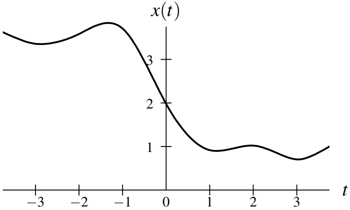
Time Shifting (Translation): Example

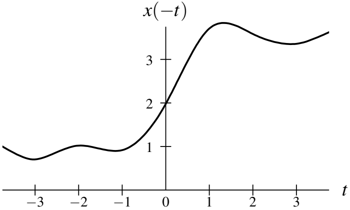


- **Time reversal** (also known as **reflection**) maps the input function x to the output function y as given by

$$y(t) = x(-t).$$

- Geometrically, the output function y is a reflection of the input function x about the (vertical) line $t = 0$.





Time Compression/Expansion (Dilation)

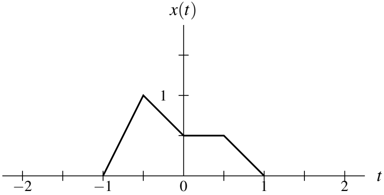
- **Time compression/expansion** (also called **dilation**) maps the input function x to the output function y as given by

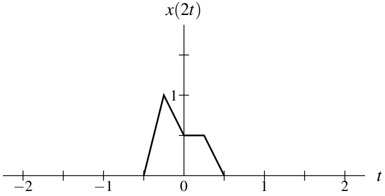
$$y(t) = x(at),$$

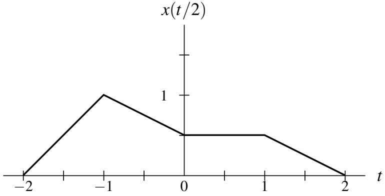
where a is a ***strictly positive*** real number.

- Such a transformation is associated with a compression/expansion along the time axis.
- If $a > 1$, y is ***compressed*** along the horizontal axis by a factor of a, relative to x.
- If $a < 1$, y is ***expanded*** (i.e., stretched) along the horizontal axis by a factor of $\frac{1}{a}$, relative to x.

Time Compression/Expansion (Dilation): Example







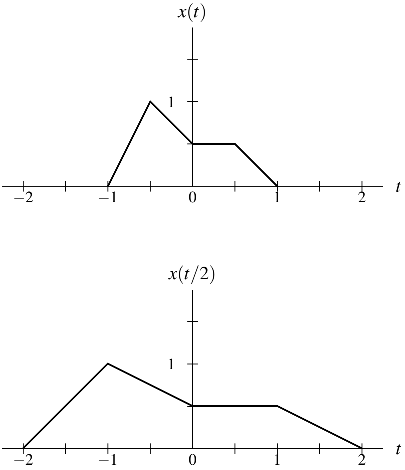
Time Scaling (Dilation/Reflection)

- **Time scaling** maps the input function x to the output function y as given by

$$y(t) = x(at),$$

where a is a ***nonzero*** real number.

- Such a transformation is associated with a dilation (i.e., compression/expansion along the time axis) and/or time reversal.
- If $|a| > 1$, the function is ***compressed*** along the time axis by a factor of $|a|$.
- If $|a| < 1$, the function is ***expanded*** (i.e., stretched) along the time axis by a factor of $\left|\frac{1}{a}\right|$.
- If $|a| = 1$, the function is neither expanded nor compressed.
- If $a < 0$, the function is also time reversed.
- Dilation (i.e., expansion/compression) and time reversal ***commute***.
- Time reversal is a special case of time scaling with $a = -1$; and time compression/expansion is a special case of time scaling with $a > 0$.


x(t)
1
−2
−1
0
1
2
t
x(2t)
1
−2
−1
0
1
2
t
x(t/2)
1
−2
−1
0
1
2
t
x(−t)
1
−2
−1
0
1
2
t

Combined Time Scaling and Time Shifting

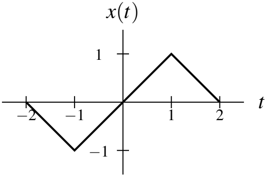
- Consider a transformation that maps the input function x to the output function y as given by

$$y(t) = x(at - b),$$

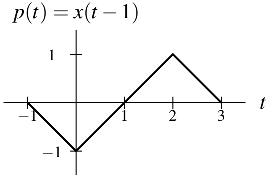
  where a and b are real numbers and $a \neq 0$.
- The above transformation can be shown to be the combination of a time-scaling operation and time-shifting operation.
- Since time scaling and time shifting ***do not commute***, we must be particularly careful about the order in which these transformations are applied.
- The above transformation has two distinct but equivalent interpretations:
  1. first, time shifting x by b, and then time scaling the result by a;
  2. first, time scaling x by a, and then time shifting the result by b/a.
- Note that the time shift is not by the same amount in both cases.
- In particular, note that when time scaling is applied first followed by time shifting, the time shift is by b/a, not b.

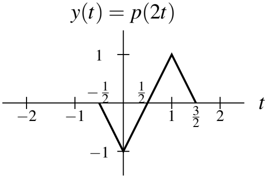
Combined Time Scaling and Time Shifting: Example

Given x as shown below, find $y(t) = x(2t-1)$.

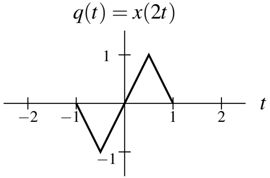


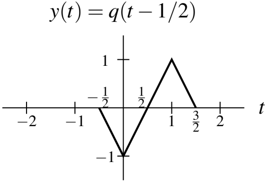
time shift by 1 and then time scale by 2





time scale by 2 and then time shift by $\frac{1}{2}$





Two Perspectives on Independent-Variable Transformations

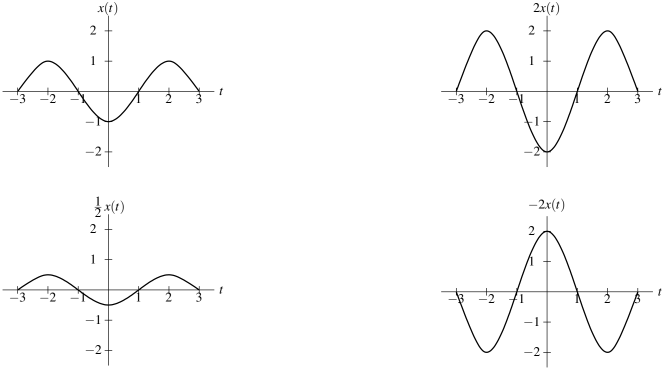
- A transformation of the independent variable can be viewed in terms of
  1. the effect that the transformation has on the ***function***; or
  2. the effect that the transformation has on the ***horizontal axis***.
- This distinction is important because such a transformation has ***opposite*** effects on the function and horizontal axis.
- For example, the (time-shifting) transformation that replaces t by $t-b$ (where b is a real number) in $x(t)$ can be viewed as a transformation that
  1. shifts the function x ***right*** by b units; or
  2. shifts the horizontal axis ***left*** by b units.
- In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the ***function***.
- If one is not careful to consider that we are interested in the function perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

- **Amplitude scaling** maps the input function x to the output function y as given by

$$y(t) = ax(t),$$

where a is a real number.

- Geometrically, the output function y is ***expanded/compressed*** in amplitude and/or ***reflected*** about the horizontal axis.

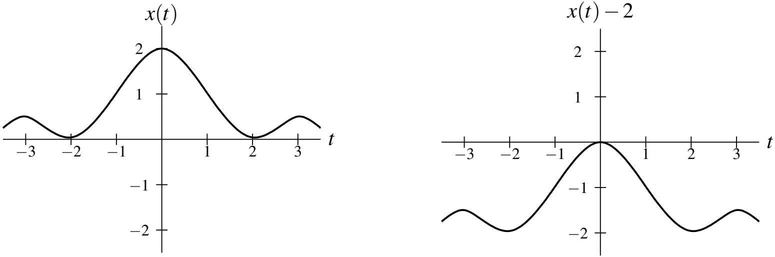


- **Amplitude shifting** maps the input function x to the output function y as given by

$$y(t) = x(t) + b,$$

where b is a real number.

- Geometrically, amplitude shifting adds a ***vertical displacement*** to x.



Combined Amplitude Scaling and Amplitude Shifting

- We can also combine amplitude scaling and amplitude shifting transformations.
- Consider a transformation that maps the input function x to the output function y, as given by

$$y(t) = ax(t) + b,$$

  where a and b are real numbers.
- Equivalently, the above transformation can be expressed as

$$y(t) = a\left[x(t) + \tfrac{b}{a}\right].$$

- The above transformation is equivalent to:
  1. first amplitude scaling x by a, and then amplitude shifting the resulting function by b; or
  2. first amplitude shifting x by b/a, and then amplitude scaling the resulting function by a.

Section 3.2

Properties of Functions

Symmetry and Addition/Multiplication

- Sums involving even and odd functions have the following properties:
  - The sum of two even functions is even.
  - The sum of two odd functions is odd.
  - The sum of an even function and odd function is neither even nor odd, provided that neither of the functions is identically zero.
- That is, the ***sum*** of functions with the ***same type of symmetry*** also has the ***same type of symmetry***.
- Products involving even and odd functions have the following properties:
  - The product of two even functions is even.
  - The product of two odd functions is even.
  - The product of an even function and an odd function is odd.
- That is, the ***product*** of functions with the ***same type of symmetry*** is ***even***, while the ***product*** of functions with ***opposite types of symmetry*** is ***odd***.

- Every function x has a ***unique*** representation of the form

$$x(t) = x_\mathrm{e}(t) + x_\mathrm{o}(t),$$

where the functions x_e and x_o are ***even*** and ***odd***, respectively.

- In particular, the functions x_e and x_o are given by

$$x_\mathrm{e}(t) = \tfrac{1}{2}\left[x(t) + x(-t)\right] \quad \text{and} \quad x_\mathrm{o}(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

- The functions x_e and x_o are called the **even part** and **odd part** of x, respectively.
- For convenience, the even and odd parts of x are often denoted as $\mathrm{Even}\{x\}$ and $\mathrm{Odd}\{x\}$, respectively.

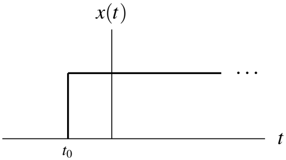
- **Sum of periodic functions.** For two periodic functions x_1 and x_2 with fundamental periods T_1 and T_2, respectively, and the sum $y = x_1 + x_2$:
  1. The sum y is periodic if and only if the ratio T_1/T_2 is a ***rational number*** (i.e., the quotient of two integers).
  2. If y is periodic, its fundamental period is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$), where $T_1/T_2 = q/r$ and q and r are integers and ***coprime*** (i.e., have no common factors). (Note that rT_1 is simply the least common multiple of T_1 and T_2.)
- Although the above theorem only directly addresses the case of the sum of two functions, the case of N functions (where $N > 2$) can be handled by applying the theorem repeatedly $N-1$ times.

Right-Sided Functions

- A function x is said to be **right sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0$$

  (i.e., x is ***only potentially nonzero to the right of*** t_0).
- An example of a right-sided function is shown below.



- A function x is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

- A causal function is a ***special case*** of a right-sided function.
- A causal function is not to be confused with a causal system. In these two contexts, the word "causal" has very different meanings.

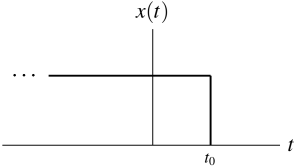
Left-Sided Functions

- A function x is said to be **left sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0$$

(i.e., x is ***only potentially nonzero to the left of*** t_0).

- An example of a left-sided function is shown below.



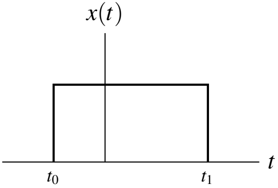
- Similarly, a function x is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

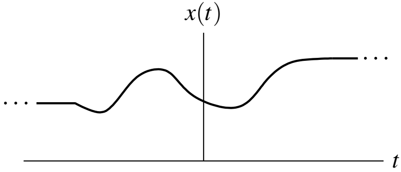
- An anticausal function is a ***special case*** of a left-sided function.
- An anticausal function is not to be confused with an anticausal system. In these two contexts, the word "anticausal" has very different meanings.

Finite-Duration and Two-Sided Functions

- A function that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite duration function is shown below.



- A function that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided function is shown below.



Bounded Functions

- A function x is said to be **bounded** if there exists some (***finite***) positive real constant A such that

$$|x(t)| \leq A \quad \text{for all } t$$

  (i.e., $x(t)$ is ***finite*** for all t).

- For example, the sine and cosine functions are bounded, since

$$|\sin t| \leq 1 \text{ for all } t \quad \text{and} \quad |\cos t| \leq 1 \text{ for all } t.$$

- In contrast, the tangent function and any nonconstant polynomial function p (e.g., $p(t) = t^2$) are unbounded, since

$$\lim_{t \to \pi/2} |\tan t| = \infty \quad \text{and} \quad \lim_{|t| \to \infty} |p(t)| = \infty.$$

Energy and Power of a Function

- The **energy** E contained in the function x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

- A signal with finite energy is said to be an **energy signal**.
- The **average power** P contained in the function x is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

- A signal with (nonzero) finite average power is said to be a **power signal**.

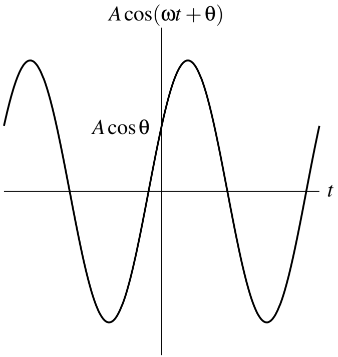
Section 3.3

Elementary Functions

- A **real sinusoidal function** is a function of the form

$$x(t) = A\cos(\omega t + \theta),$$

  where A, ω, and θ are ***real*** constants.
- Such a function is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.
- A real sinusoid has a plot resembling that shown below.



Complex Exponential Functions

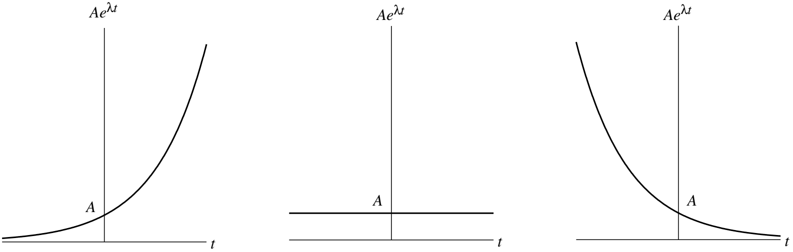
- A **complex exponential function** is a function of the form

$$x(t) = Ae^{\lambda t},$$

where A and λ are ***complex*** constants.

- A complex exponential can exhibit one of a number of ***distinct modes of behavior***, depending on the values of its parameters A and λ.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

- A **real exponential function** is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be ***real*** numbers.
- A real exponential can exhibit one of ***three distinct modes*** of behavior, depending on the value of λ, as illustrated below.
- If $\lambda > 0$, $x(t)$ ***increases*** exponentially as t increases (i.e., a growing exponential).
- If $\lambda < 0$, $x(t)$ ***decreases*** exponentially as t increases (i.e., a decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the ***constant*** A.



$\lambda > 0$ $\lambda = 0$ $\lambda < 0$

Complex Sinusoidal Functions

- A complex sinusoidal function is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is ***complex*** and λ is ***purely imaginary*** (i.e., $\mathrm{Re}\{\lambda\} = 0$).
- That is, a **complex sinusoidal function** is a function of the form

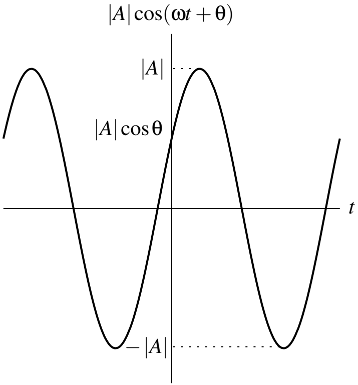
$$x(t) = Ae^{j\omega t},$$

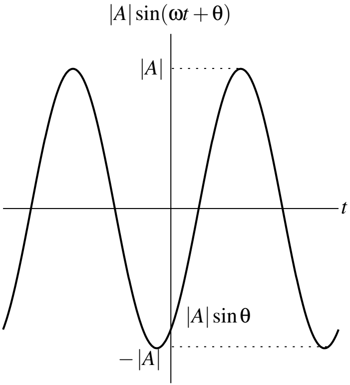
  where A is ***complex*** and ω is ***real***.
- By expressing A in polar form as $A = |A|\,e^{j\theta}$ (where θ is real) and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are the same except for a time shift.
- Also, x is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.

- The graphs of $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ have the forms shown below.





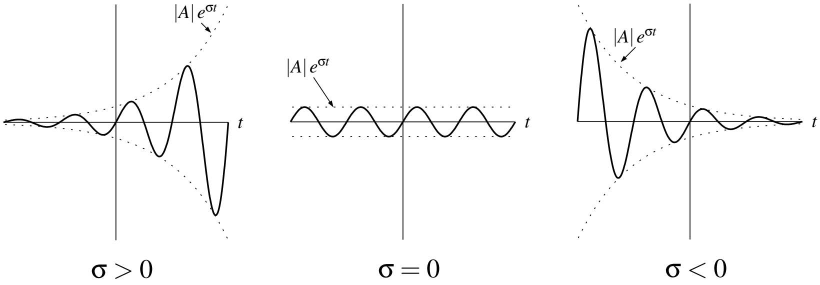
General Complex Exponential Functions

- In the most general case of a complex exponential function $x(t) = Ae^{\lambda t}$, A and λ are both ***complex***.
- Letting $A = |A| e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A| e^{\sigma t} \cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j \underbrace{|A| e^{\sigma t} \sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real exponential and real sinusoid.
- One of ***three distinct modes*** of behavior is exhibited by $x(t)$, depending on the value of σ.
- If $\sigma = 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are ***real sinusoids***.
- If $\sigma > 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a growing real exponential***.
- If $\sigma < 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a decaying real exponential***.

- The ***three modes of behavior*** for $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are illustrated below.



Relationship Between Complex Exponentials and Real Sinusoids

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$Ae^{j\omega t} = A\cos\omega t + jA\sin\omega t.$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

$$A\cos(\omega t+\theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \quad \text{and}$$
$$A\sin(\omega t+\theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right].$$

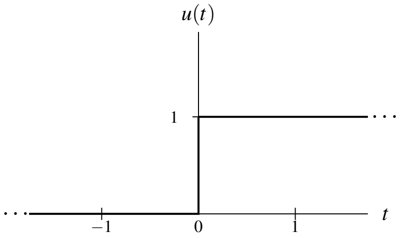
- Note that, above, we are simply ***restating results*** from the (appendix) material on complex analysis.

Unit-Step Function

- The **unit-step function** (also known as the **Heaviside function**), denoted u, is defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which u is used in practice, the actual ***value of $u(0)$*** is unimportant. Sometimes values of 0 and $\frac{1}{2}$ are also used for $u(0)$.
- A plot of this function is shown below.

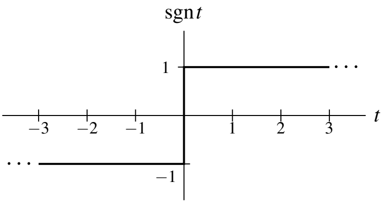


Signum Function

- The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases}$$

- From its definition, one can see that the signum function simply computes the ***sign*** of a number.
- A plot of this function is shown below.

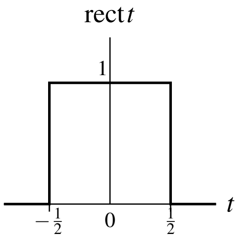


Rectangular Function

- The **rectangular function** (also called the unit-rectangular pulse function), denoted rect, is given by

$$\operatorname{rect} t = \begin{cases} 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual ***value of*** $\operatorname{rect} t$ ***at*** $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.
- A plot of this function is shown below.

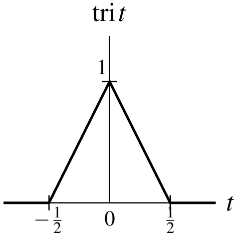


Triangular Function

- The **triangular function** (also called the unit-triangular pulse function), denoted tri, is defined as

$$\operatorname{tri} t = \begin{cases} 1 - 2\left|t\right| & \left|t\right| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

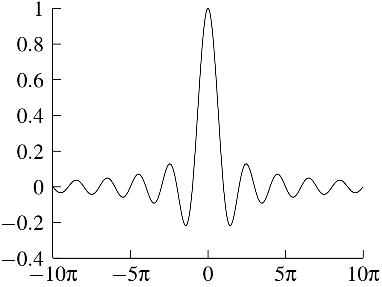
- A plot of this function is shown below.



- The **cardinal sine** function, denoted sinc, is given by

$$\operatorname{sinc} t = \frac{\sin t}{t}.$$

- By l'Hopital's rule, $\operatorname{sinc} 0 = 1$.
- A plot of this function for part of the real line is shown below. [Note that the oscillations in $\operatorname{sinc} t$ do not die out for finite t.]

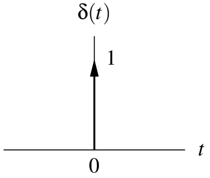


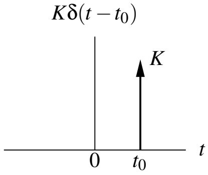
Unit-Impulse Function

- The **unit-impulse function** (also known as the **Dirac delta function** or **delta function**), denoted δ, is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a ***generalized function***. Consequently, the δ function sometimes behaves in unusual ways.
- Graphically, the delta function is represented as shown below.



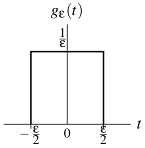


Unit-Impulse Function as a Limit

- Define

$$g_\varepsilon(t) = \begin{cases} 1/\varepsilon & |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$

- The function g_ε has a plot of the form shown below.



- Clearly, for any choice of ε, $\int_{-\infty}^{\infty} g_\varepsilon(t)dt = 1$.
- The function δ can be obtained as the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$

- That is, δ can be viewed as a ***limiting case of a rectangular pulse*** where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

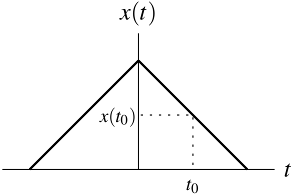
- **Equivalence property.** For any continuous function x and any real constant t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

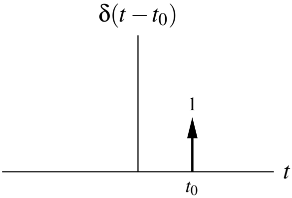
- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$

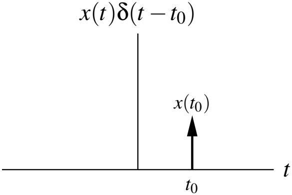
Graphical Interpretation of Equivalence Property



Function x



Time-Shifted Unit-Impulse Function



Product

- For real constants a and b where $a \leq b$, consider a function x of the form

$$x(t) = \begin{cases} 1 & a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

  (i.e., x is a ***rectangular pulse*** of height one, with a ***rising edge at a*** and ***falling edge at b***).

- The function x can be equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

  (i.e., the difference of two time-shifted unit-step functions).

- Unlike the original expression for x, this latter expression for x ***does not involve multiple cases***.
- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

- The idea from the previous slide can be extended to handle any function that is defined in a ***piecewise manner*** (i.e., via an expression involving multiple cases).
- That is, by using unit-step functions, we can always collapse a formula involving multiple cases into a single expression.
- Often, simplifying a formula in this way can be quite beneficial.

Section 3.4

Continuous-Time (CT) Systems

CT Systems

- A system with input x and output y can be described by the equation

$$y = \mathcal{H}x,$$

  where $\mathcal{H}$ denotes an operator (i.e., transformation).
- Note that the operator $\mathcal{H}$ ***maps a function to a function*** (not a number to a number).
- Alternatively, we can express the above relationship using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

- If clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

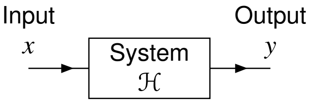
$$x \to y.$$

- Note that the symbols "$\to$" and "=" have ***very different*** meanings.
- The symbol "$\to$" should be read as "***produces***" (not as "equals").

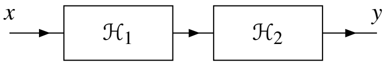
Remarks on Operator Notation for Systems

- For a system operator $\mathcal{H}$ and a function x, $\mathcal{H}x$ is the function produced as the output of the system $\mathcal{H}$ when the input is the function x.
- Brackets around the operand of an operator are ***usually omitted when not required*** for grouping.
- For example, for an operator $\mathcal{H}$, a function x, and a real number t, we would normally prefer to write:
  1. $\mathcal{H}x$ instead of the equivalent expression $\mathcal{H}(x)$; and
  2. $\mathcal{H}x(t)$ instead of the equivalent expression $\mathcal{H}(x)(t)$.
- Also, note that $\mathcal{H}x$ is a ***function*** and $\mathcal{H}x(t)$ is a ***number*** (namely, the value of the function $\mathcal{H}x$ evaluated at the point t).
- In the expression $\mathcal{H}(x_1 + x_2)$, the brackets are needed for grouping, since $\mathcal{H}(x_1 + x_2) \not\equiv \mathcal{H}x_1 + x_2$ (where "$\not\equiv$" means "not equivalent").
- When multiple operators are applied, they group from ***right to left***.
- For example, for the operators $\mathcal{H}_1$ and $\mathcal{H}_2$, and the function x, the expression $\mathcal{H}_2\mathcal{H}_1x$ means $\mathcal{H}_2[\mathcal{H}_1(x)]$.

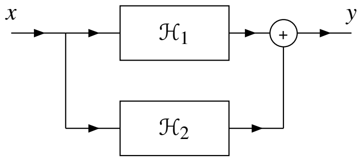
- Often, a system defined by the operator $\mathcal{H}$ and having the input x and output y is represented in the form of a ***block diagram*** as shown below.



- ***Two basic ways*** in which systems can be interconnected are shown below.



Series



Parallel

- A **series** (or **cascade**) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = \mathcal{H}_2\mathcal{H}_1 x.$$

- A **parallel** connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x.$$

Section 3.5

Properties of (CT) Systems

Memory

- A system $\mathcal{H}$ is said to be **memoryless** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t \neq t_0$.
- In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the ***same*** point in time.
- A system that is not memoryless is said to have **memory**.
- Although simple, a memoryless system is ***not very flexible***, since its current output value cannot rely on past or future values of the input.

Causality

- A system $\mathcal{H}$ is said to be **causal** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t > t_0$.
- In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the ***same or earlier points*** in time (i.e., ***not later points in time***).
- If the independent variable t represents time, a system must be causal in order to be ***physically realizable***.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable ***need not always represent time*** (e.g., the independent variable might represent position).
- A memoryless system is always causal, although the converse is not necessarily true.

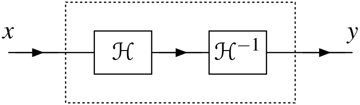
Invertibility

- The **inverse** of a system $\mathcal{H}$ (if it exists) is another system $\mathcal{H}^{-1}$ such that, for every function x,

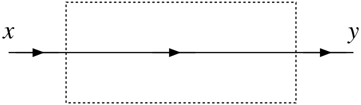
$$\mathcal{H}^{-1}\mathcal{H}x = x$$

  (i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{H}^{-1}$ is a system whose input and output are equal).
- A system is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input can always be ***uniquely*** determined from its output.
- An invertible system will always produce ***distinct outputs*** from any two ***distinct inputs***.
- To show that a system is ***invertible***, we simply find the ***inverse system***.
- To show that a system is ***not invertible***, we find ***two distinct inputs*** that result in ***identical outputs***.
- In practical terms, invertible systems are "nice" in the sense that their ***effects can be undone***.

- A system $\mathcal{H}^{-1}$ being the inverse of $\mathcal{H}$ means that the following two systems are equivalent (i.e., $\mathcal{H}^{-1}\mathcal{H}$ ***is an identity***):



System 1: $y = \mathcal{H}^{-1}\mathcal{H}x$



System 2: $y = x$

Bounded-Input Bounded-Output (BIBO) Stability

- A system $\mathcal{H}$ is **BIBO stable** if, for every bounded function x, $\mathcal{H}x$ is bounded (i.e., $|x(t)| < \infty$ for all t implies that $|\mathcal{H}x(t)| < \infty$ for all t).
- In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.
- To show that a system is ***BIBO stable***, we must show that ***every bounded input*** leads to a ***bounded output***.
- To show that a system is ***not BIBO stable***, we only need to find a single ***bounded input*** that leads to an ***unbounded output***.
- In practical terms, a BIBO stable system is ***well behaved*** in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time.
- Usually, a system that is not BIBO stable will have ***serious safety issues***.
- For example, a portable music player with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized human (and likely a big lawsuit as well).

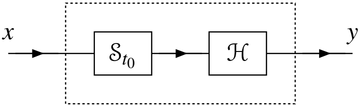
Time Invariance (TI)

- A system $\mathcal{H}$ is said to be **time invariant (TI)** if, for every function x and every real constant t_0, the following condition holds:

$$\mathcal{H}x(t-t_0) = \mathcal{H}x'(t) \text{ for all } t, \quad \text{where} \quad x'(t) = x(t-t_0)$$

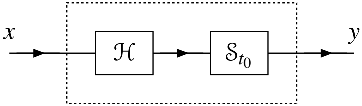
  (i.e., $\mathcal{H}$ ***commutes with time shifts***).
- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an ***identical time shift*** in the output.
- A system that is not time invariant is said to be **time varying**.
- In simple terms, a time invariant system is a system whose behavior ***does not change*** with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much ***easier to design and analyze***, since their behavior does not change with respect to time.

- Let $\mathcal{S}_{t_0}$ denote an operator that applies a ***time shift of t_0*** to a function (i.e., $\mathcal{S}_{t_0}x(t) = x(t-t_0)$).
- A system $\mathcal{H}$ is ***time invariant*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with*** $\mathcal{S}_{t_0}$):



System 1: $y = \mathcal{H}\mathcal{S}_{t_0}x$

$$\begin{bmatrix} y(t) = \mathcal{H}x'(t) \\ x'(t) = \mathcal{S}_{t_0}x(t) = x(t-t_0) \end{bmatrix}$$



System 2: $y = \mathcal{S}_{t_0}\mathcal{H}x$

$$\left[y(t) = \mathcal{H}x(t-t_0)\right]$$

- A system $\mathcal{H}$ is said to be **additive** if, for all functions x_1 and x_2, the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

(i.e., ***$\mathcal{H}$ commutes with addition***).

- A system $\mathcal{H}$ is said to be **homogeneous** if, for every function x and every complex constant a, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

(i.e., ***$\mathcal{H}$ commutes with scalar multiplication***).

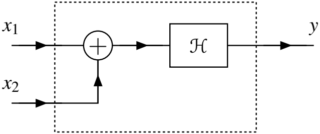
- A system that is both additive and homogeneous is said to be **linear**.
- In other words, a system $\mathcal{H}$ is ***linear***, if for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

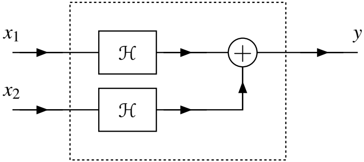
(i.e., ***$\mathcal{H}$ commutes with linear combinations***).

- The linearity property is also referred to as the **superposition** property.
- Practically speaking, linear systems are much ***easier to design and analyze*** than nonlinear systems.

- The system $\mathcal{H}$ is ***additive*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with addition***):

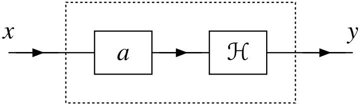


System 1: $y = \mathcal{H}(x_1 + x_2)$

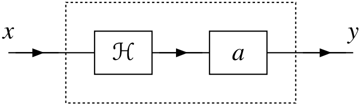


System 2: $y = \mathcal{H}x_1 + \mathcal{H}x_2$

- The system $\mathcal{H}$ is ***homogeneous*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with scalar multiplication***):

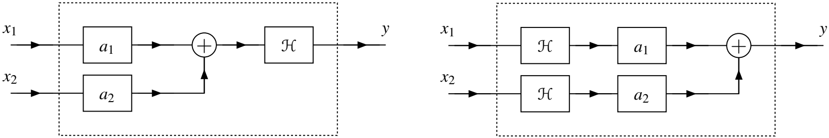


System 1: $y = \mathcal{H}(ax)$



System 2: $y = a\mathcal{H}x$

- The system $\mathcal{H}$ is ***linear*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with linear combinations***):



System 1: $y = \mathcal{H}(a_1x_1 + a_2x_2)$ System 2: $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$

- A function x is said to be an **eigenfunction** of the system $\mathcal{H}$ with the **eigenvalue** λ if

$$\mathcal{H}x = \lambda x,$$

where λ is a complex constant.

- In other words, the system $\mathcal{H}$ acts as an ideal amplifier for each of its eigenfunctions x, where the amplifier gain is given by the corresponding eigenvalue λ.
- Different systems have different eigenfunctions.
- Many of the mathematical tools developed for the study of CT systems have eigenfunctions as their basis.

Part 4

Continuous-Time Linear Time-Invariant (LTI) Systems

Why Linear Time-Invariant (LTI) Systems?

- In engineering, linear time-invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Section 4.1

Convolution

CT Convolution

- The (CT) **convolution** of the functions x and h, denoted $x * h$, is defined as the function

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- The convolution result $x * h$ evaluated at the point t is simply a weighted average of the function x, where the weighting is given by h time reversed and shifted by t.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in systems theory.
- In particular, convolution has a special significance in the context of LTI systems.

Practical Convolution Computation

- To compute the convolution

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau,$$

we proceed as follows:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $x * h(t)$.
4. Increase t gradually until the expression for $x * h(t)$ changes form. Record the interval over which the expression for $x * h(t)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.
6. The results for the various intervals can be combined in order to obtain an expression for $x * h(t)$ for all t.

- The convolution operation is ***commutative***. That is, for any two functions x and h,

$$x * h = h * x.$$

- The convolution operation is ***associative***. That is, for any functions x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is ***distributive*** with respect to addition. That is, for any functions x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

- For any function x,

$$x * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t).$$

- Thus, any function x can be written in terms of an expression involving δ.
- Moreover, δ is the ***convolutional identity***. That is, for any function x,

$$x * \delta = x.$$

Periodic Convolution

- The convolution of two periodic functions is usually not well defined.
- This motivates an alternative notion of convolution for periodic functions known as periodic convolution.
- The **periodic convolution** of the T-periodic functions x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t-\tau)d\tau,$$

  where $\int_T$ denotes integration over an interval of length T.
- The periodic convolution and (linear) convolution of the T-periodic functions x and h are related as follows:

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

  (i.e., $x_0(t)$ equals $x(t)$ over a single period of x and is zero elsewhere).

Section 4.2

Convolution and LTI Systems

- The response h of a system $\mathcal{H}$ to the input δ is called the **impulse response** of the system (i.e., $h = \mathcal{H}\delta$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

- In other words, a LTI system simply ***computes a convolution***.
- Furthermore, a LTI system is ***completely characterized*** by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.
- Since the impulse response of a LTI system is an extremely useful quantity, we often want to determine this quantity in a practical setting.
- Unfortunately, in practice, the impulse response of a system cannot be determined directly from the definition of the impulse response.

Step Response

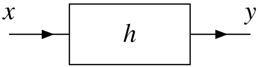
- The response s of a system $\mathcal{H}$ to the input u is called the **step response** of the system (i.e., $s = \mathcal{H}u$).
- The impulse response h and step response s of a system are related as

$$h(t) = \frac{ds(t)}{dt}.$$

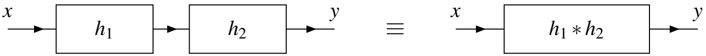
- Therefore, the impulse response of a system can be determined from its step response by differentiation.
- The step response provides a practical means for determining the impulse response of a system.

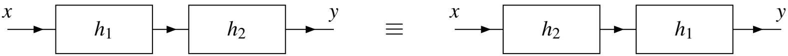
Block Diagram Representation of LTI Systems

- Often, it is convenient to represent a (CT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.

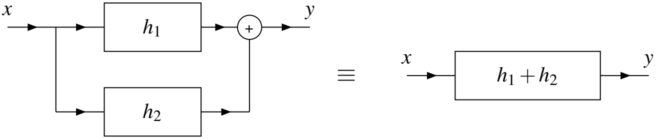


- The ***series*** interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h = h_1 * h_2$. That is, we have the equivalences shown below.





- The ***parallel*** interconnection of the LTI systems with impulse responses h_1 and h_2 is a LTI system with the impulse response $h = h_1 + h_2$. That is, we have the equivalence shown below.



Section 4.3

Properties of LTI Systems

- A LTI system with impulse response h is memoryless if and only if

$$h(t) = 0 \quad \text{for all } t \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(t) = K\delta(t),$$

where K is a complex constant.

- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\delta) = Kx$$

(i.e., the system is an ideal amplifier).

- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

Causality

- A LTI system with impulse response h is causal if and only if

$$h(t) = 0 \quad \text{for all } t < 0$$

  (i.e., h is a causal function).
- It is due to the above relationship that we call a function x, satisfying

$$x(t) = 0 \quad \text{for all } t < 0,$$

  a causal function.

Invertibility

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{\text{inv}} = \delta.$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a function h_{inv} such that

$$h * h_{\text{inv}} = \delta.$$

- Except in simple cases, the above condition is often quite difficult to test.

- A LTI system with impulse response h is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

(i.e., h is ***absolutely integrable***).

Eigenfunctions of LTI Systems

- As it turns out, every complex exponential is an eigenfunction of all LTI systems.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{e^{st}\}(t) = H(s)e^{st},$$

  where s is a complex constant and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

- That is, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.
- We refer to H as the **system function** (or **transfer function**) of the system $\mathcal{H}$.
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$.

Representations of Functions Using Eigenfunctions

- Consider a LTI system with input x, output y, and system function H.
- Suppose that the input x can be expressed as the linear combination of complex exponentials

$$x(t) = \sum_k a_k e^{s_k t},$$

where the a_k and s_k are complex constants.

- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$

- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as a linear combination of the ***same*** complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Part 5

Continuous-Time Fourier Series (CTFS)

- The (CT) Fourier series is a representation for ***periodic*** functions.
- With a Fourier series, a function is represented as a ***linear combination of complex sinusoids***.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- For example, complex sinusoids are continuous and differentiable. They are also easy to integrate and differentiate.
- Perhaps, most importantly, complex sinusoids are ***eigenfunctions*** of LTI systems.

Section 5.1

Fourier Series

- A set of complex sinusoids is said to be **harmonically related** if there exists some constant ω_0 such that the fundamental frequency of each complex sinusoid is an integer multiple of ω_0.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(t) = e^{jk\omega_0 t} \quad \text{for all integer } k.$$

- The fundamental frequency of the kth complex sinusoid ϕ_k is $k\omega_0$, an integer multiple of ω_0.
- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic.
- More specifically, a linear combination of these complex sinusoids is periodic with period $T = \frac{2\pi}{\omega_0}$.

CT Fourier Series

- A periodic complex function x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Such a representation is known as (the complex exponential form of) a (CT) **Fourier series**, and the c_k are called **Fourier series coefficients**.
- The above formula for x is often referred to as the **Fourier series synthesis equation**.
- The terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K\omega_0$.
- To denote that a function x has the Fourier series coefficient sequence c_k, we write

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k.$$

- The periodic function x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ has the Fourier series coefficients c_k given by

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt,$$

where $\int_T$ denotes integration over an arbitrary interval of length T (i.e., one period of x).

- The above equation for c_k is often referred to as the **Fourier series analysis equation**.

Trigonometric Forms of a Fourier Series

- Consider the periodic function x with the Fourier series coefficients c_k.
- If x is real, then its Fourier series can be rewritten in two other forms, known as the combined trigonometric and trigonometric forms.
- The **combined trigonometric form** of a Fourier series has the appearance

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k),$$

  where $\theta_k = \arg c_k$.
- The **trigonometric form** of a Fourier series has the appearance

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t\right],$$

  where $\alpha_k = 2\operatorname{Re} c_k$ and $\beta_k = -2\operatorname{Im} c_k$.
- Note that the trigonometric forms contain only ***real*** quantities.

Section 5.2

Convergence Properties of Fourier Series

- Since a Fourier series can have an infinite number of terms, and an infinite sum may or may not converge, we need to consider the issue of convergence.
- That is, when we claim that a periodic function x is equal to the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, is this claim actually correct?
- Consider a periodic function x that we wish to represent with the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Let x_N denote the Fourier series truncated after the Nth harmonic components as given by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

- Here, we are interested in whether $\lim_{N\to\infty} x_N$ is equal (in some sense) to x.

Convergence of Fourier Series (Continued)

- The ***error*** in approximating $x(t)$ by $x_N(t)$ is given by

$$e_N(t) = x(t) - x_N(t),$$

and the corresponding ***mean-squared error (MSE)*** (i.e., energy of the error) is given by

$$E_N = \frac{1}{T}\int_T |e_N(t)|^2\,dt.$$

- If $\lim_{N\to\infty} e_N(t) = 0$ for all t (i.e., the error goes to zero at every point), the Fourier series is said to converge **pointwise** to $x(t)$.
- If convergence is pointwise and the rate of convergence is the same everywhere, the convergence is said to be **uniform**.
- If $\lim_{N\to\infty} E_N = 0$ (i.e., the energy of the error goes to zero), the Fourier series is said to converge to x in the **MSE** sense.
- Pointwise convergence implies MSE convergence, but the converse is not true. Thus, pointwise convergence is a much stronger condition than MSE convergence.

Convergence of Fourier Series: Continuous Case

- If a periodic function x is ***continuous*** and its Fourier series coefficients c_k are ***absolutely summable*** (i.e., $\sum_{k=-\infty}^{\infty} |c_k| < \infty$), then the Fourier series representation of x converges ***uniformly*** (i.e., pointwise at the same rate everywhere).
- Since, in practice, we often encounter functions with discontinuities (e.g., a square wave), the above result is of somewhat limited value.

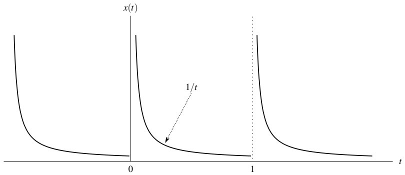
- If a periodic function x has ***finite energy*** in a single period (i.e., $\int_T |x(t)|^2 dt < \infty$), the Fourier series converges in the ***MSE*** sense.
- Since, in situations of practice interest, the finite-energy condition in the above theorem is typically satisfied, the theorem is usually applicable.
- It is important to note, however, that MSE convergence (i.e., $E = 0$) does not necessarily imply pointwise convergence (i.e., $\tilde{x}(t) = x(t)$ for all t).
- Thus, the above convergence theorem does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier series converges.

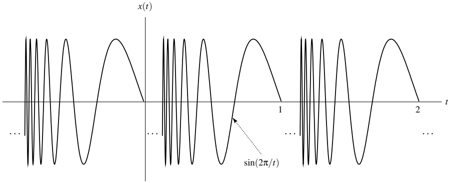
- The **Dirichlet conditions** for the periodic function x are as follows:
  1. Over a single period, x is ***absolutely integrable*** (i.e., $\int_T |x(t)|\,dt < \infty$).
  2. Over a single period, x has a finite number of maxima and minima (i.e., x is of ***bounded variation***).
  3. Over any finite interval, x has a ***finite number of discontinuities***, each of which is ***finite***.
- If a periodic function x satisfies the ***Dirichlet conditions***, then:
  1. The Fourier series converges pointwise everywhere to x, except at the points of discontinuity of x.
  2. At each point t_a of discontinuity of x, the Fourier series $\tilde{x}$ converges to

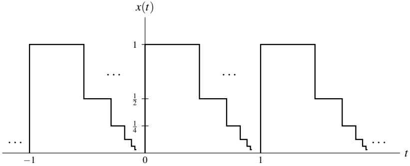
$$\tilde{x}(t_a) = \tfrac{1}{2}\left[x(t_a^-) + x(t_a^+)\right],$$

     where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.
- Since most functions tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier series at every point, this result is often very useful in practice.

Examples of Functions Violating the Dirichlet Conditions



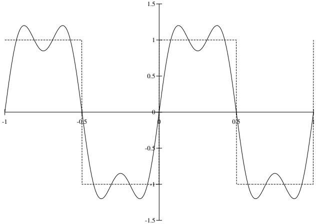




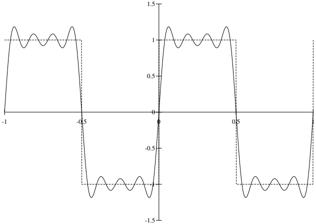
Gibbs Phenomenon

- In practice, we frequently encounter functions with discontinuities.
- When a function x has discontinuities, the Fourier series representation of x does not converge uniformly (i.e., at the same rate everywhere).
- The rate of convergence is much slower at points in the vicinity of a discontinuity.
- Furthermore, in the vicinity of a discontinuity, the truncated Fourier series x_N exhibits ripples, where the peak amplitude of the ripples does not seem to decrease with increasing N.
- As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N, the peak amplitude of the ripples remains approximately constant.
- This behavior is known as **Gibbs phenomenon**.
- The above behavior is one of the weaknesses of Fourier series (i.e., Fourier series converge very slowly near discontinuities).

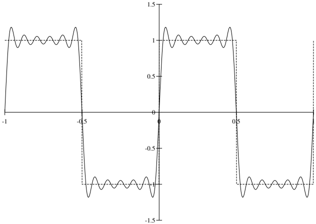
Gibbs Phenomenon: Periodic Square Wave Example



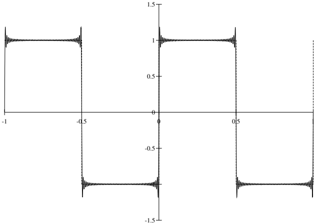
Fourier series truncated after the 3rd harmonic components



Fourier series truncated after the 7th harmonic components



Fourier series truncated after the 11th harmonic components



Fourier series truncated after the 101st harmonic components

Section 5.3

Properties of Fourier Series

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Modulation	$e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a^*_{-k}
Periodic Convolution	$x \circledast y(t)$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

Property	
Parseval's Relation	$\frac{1}{T} \int_T \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

Linearity

- Let x and y be two periodic functions with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$\alpha x(t) + \beta y(t) \overset{\text{CTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$

  where α and β are complex constants.
- That is, a linear combination of functions produces the same linear combination of their Fourier series coefficients.

Time Shifting (Translation)

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(t - t_0) \overset{\text{CTFS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} c_k = e^{-jk(2\pi/T)t_0} c_k,$$

  where t_0 is a real constant.
- In other words, time shifting a periodic function changes the argument (but not magnitude) of its Fourier series coefficients.

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$e^{jM(2\pi/T)t}x(t) = e^{jM\omega_0 t}x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_{k-M},$$

  where M is an integer constant.
- In other words, multiplying a periodic function by $e^{jM\omega_0 t}$ shifts the Fourier-series coefficient sequence.

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \stackrel{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(-t) \stackrel{\text{CTFS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversal of a function results in a time reversal of its Fourier series coefficients.

Conjugation

- For a T-periodic function x with Fourier series coefficient sequence c, the following property holds:

$$x^*(t) \overset{\text{CTFS}}{\longleftrightarrow} c^*_{-k}$$

- In other words, conjugating a function has the effect of time reversing and conjugating the Fourier series coefficient sequence.

Periodic Convolution

- Let x and y be two periodic functions with the same period T. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$x \circledast y(t) \overset{\text{CTFS}}{\longleftrightarrow} Ta_k b_k.$$

- In other words, periodic convolution of two functions corresponds to the multiplication (up to a scale factor) of their Fourier-series coefficient sequences.

- Let x and y be two periodic functions with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$x(t)y(t) \overset{\text{CTFS}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

- As we shall see later, the above summation is the DT convolution of a and b.
- In other words, the multiplication of two periodic functions corresponds to the DT convolution of their corresponding Fourier-series coefficient sequences.

Even and Odd Symmetry

- For a T-periodic function x with Fourier series coefficient sequence c, the following properties hold:

$$x \text{ is even} \Leftrightarrow c \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow c \text{ is odd}.$$

- In other words, the even/odd symmetry properties of x and c always match.

Real Functions

- A function x is ***real*** if and only if its Fourier series coefficient sequence c satisfies

$$c_k = c^*_{-k} \text{ for all } k$$

(i.e., c is ***conjugate symmetric***).

- Thus, for a real-valued function, the negative-indexed Fourier series coefficients are ***redundant***, as they are completely determined by the nonnegative-indexed coefficients.
- From properties of complex numbers, one can show that $c_k = c^*_{-k}$ is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}$$

(i.e., $|c_k|$ is ***even*** and $\arg c_k$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that c is real.

- A function x and its Fourier series coefficient sequence a satisfy the following relationship:

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2 .$$

- The above relationship is simply stating that the amount of energy in x (i.e., $\frac{1}{T}\int_T |x(t)|^2\,dt$) and the amount of energy in the Fourier series coefficient sequence a (i.e., $\sum_{k=-\infty}^{\infty} |a_k|^2$) are equal.
- In other words, the transformation between a function and its Fourier series coefficient sequence preserves energy.

Other Properties of Fourier Series

- For a T-periodic function x with Fourier-series coefficient sequence c, the following properties hold:
    1. c_0 is the average value of x over a single period;
    2. x is real and even $\Leftrightarrow$ c is real and even; and
    3. x is real and odd $\Leftrightarrow$ c is purely imaginary and odd.

Section 5.4

Fourier Series and Frequency Spectra

A New Perspective on Functions: The Frequency Domain

- The Fourier series provides us with an entirely new way to view functions.
- Instead of viewing a function as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a function as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- This so called frequency-domain perspective is of fundamental importance in engineering.
- Many engineering problems can be solved ***much more easily*** using the frequency domain than the time domain.
- The Fourier series coefficients of a function x provide a means to ***quantify*** how much information x has at different frequencies.
- The distribution of information in a function over different frequencies is referred to as the ***frequency spectrum*** of the function.

- To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the function x, it is helpful to write the Fourier series with the c_k expressed in ***polar form*** as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} |c_k| \, e^{j(k\omega_0 t + \arg c_k)}.$$

- Clearly, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has been ***amplitude scaled*** by a factor of $|c_k|$ and ***time-shifted*** by an amount that depends on $\arg c_k$.
- For a given k, the ***larger*** $|c_k|$ is, the larger is the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the ***larger the contribution*** the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation.
- In this way, we can use $|c_k|$ as a ***measure*** of how much information a function x has at the frequency $k\omega_0$.

- The Fourier series coefficients c_k are referred to as the **frequency spectrum** of x.
- The magnitudes $|c_k|$ of the Fourier series coefficients are referred to as the **magnitude spectrum** of x.
- The arguments $\arg c_k$ of the Fourier series coefficients are referred to as the **phase spectrum** of x.
- Normally, the spectrum of a function is plotted against frequency $k\omega_0$ instead of k.
- Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, the frequency spectrum is ***discrete*** in the independent variable (i.e., frequency).
- Due to the general appearance of frequency-spectrum plot (i.e., a number of vertical lines at various frequencies), we refer to such spectra as **line spectra**.

Frequency Spectra of Real Functions

- Recall that, for a real function x, the Fourier series coefficient sequence c satisfies

$$c_k = c_{-k}^*$$

(i.e., c is ***conjugate symmetric***), which is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}.$$

- Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real function is always ***even***.
- Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real function is always ***odd***.
- Due to the symmetry in the frequency spectra of real functions, we typically ***ignore negative frequencies*** when dealing with such functions.
- In the case of functions that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

Section 5.5

Fourier Series and LTI Systems

Frequency Response

- Recall that a LTI system $\mathcal{H}$ with impulse response h is such that $\mathcal{H}\{e^{st}\}(t) = H(s)e^{st}$, where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$. (That is, complex exponentials are ***eigenfunctions*** of LTI systems.)
- Since a complex sinusoid is a ***special case*** of a complex exponential, we can reuse the above result for the special case of complex sinusoids.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{e^{j\omega t}\}(t) = H(j\omega)e^{j\omega t},$$

  where ω is a real constant and

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

- That is, $e^{j\omega t}$ is an ***eigenfunction*** of a LTI system and $H(j\omega)$ is the corresponding ***eigenvalue***.
- We refer to $H(j\omega)$ as the **frequency response** of the system $\mathcal{H}$.

Fourier Series and LTI Systems

- Consider a LTI system with input x, output y, and frequency response $H(j\omega)$.
- Suppose that the T-periodic input x is expressed as the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{where } \omega_0 = \tfrac{2\pi}{T}.$$

- Using our knowledge about the ***eigenfunctions*** of LTI systems, we can conclude

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t}.$$

- Thus, if the input x to a LTI system is a Fourier series, the output y is also a Fourier series. More specifically, if $x(t) \stackrel{\text{CTFS}}{\longleftrightarrow} c_k$ then $y(t) \stackrel{\text{CTFS}}{\longleftrightarrow} H(jk\omega_0)c_k$.
- The above formula can be used to determine the output of a LTI system from its input in a way that ***does not require convolution***.

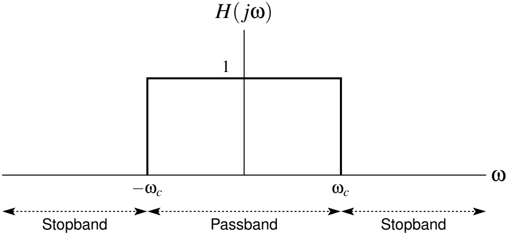
Filtering

- In many applications, we want to ***modify the spectrum*** of a function by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a function is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

  where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

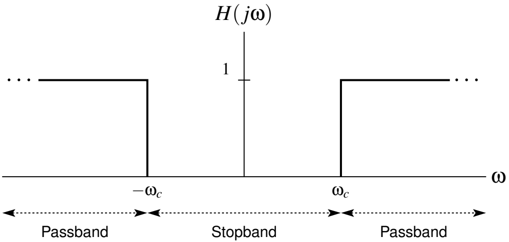


Ideal Highpass Filter

- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(j\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

  where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

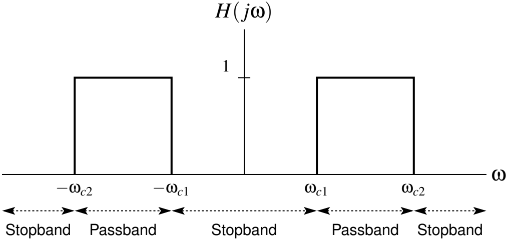


Ideal Bandpass Filter

- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(j\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.



Part 6

Continuous-Time Fourier Transform (CTFT)

- The (CT) Fourier series provide an extremely useful representation for periodic functions.
- Often, however, we need to deal with functions that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The (CT) Fourier transform can be used to represent both periodic and aperiodic functions.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Section 6.1

Fourier Transform

- The Fourier series is an extremely useful function representation.
- Unfortunately, this function representation can only be used for periodic functions, since a Fourier series is inherently periodic.
- Many functions are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can be applied to aperiodic functions.
- By viewing an aperiodic function as the limiting case of a periodic function with period T where $T \to \infty$, we can use the Fourier series to develop a function representation that can be used for aperiodic functions, known as the Fourier transform.

- Recall that the Fourier series representation of a T-periodic function x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk(2\pi/T)t} dt \right)}_{c_k} e^{jk(2\pi/T)t}.$$

- In the above representation, if we take the limit as $T \to \infty$, we obtain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right)}_{X(\omega)} e^{j\omega t} d\omega$$

(i.e., as $T \to \infty$, the outer summation becomes an integral, $\frac{1}{T} = \frac{\omega_0}{2\pi}$ becomes $\frac{1}{2\pi} d\omega$, and $k(2\pi/T) = k\omega_0$ becomes ω).

- This more general function representation is known as the Fourier transform representation.

- The classical Fourier transform for aperiodic functions does not exist for some functions of great practical interest, such as:
  - a nonzero constant function;
  - a periodic function (e.g., a real or complex sinusoid);
  - the unit-step function (i.e., u); and
  - the signum function (i.e., sgn).
- Fortunately, the Fourier transform can be extended to handle such functions, resulting in what is known as the **generalized Fourier transform**.
- For our purposes, we can think of the classical and generalized Fourier transforms as being defined by the same formulas.
- Therefore, in what follows, we will not typically make a distinction between the classical and generalized Fourier transforms.

- The (CT) **Fourier transform** of the function x, denoted $\mathcal{F}x$ or X, is given by

$$\mathcal{F}x(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

- The preceding equation is sometimes referred to as **Fourier transform analysis equation** (or **forward Fourier transform equation**).
- The **inverse Fourier transform** of X, denoted $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(t) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega.$$

- The preceding equation is sometimes referred to as the **Fourier transform synthesis equation** (or **inverse Fourier transform equation**).
- As a matter of notation, to denote that a function x has the Fourier transform X, we write $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$.
- A function x and its Fourier transform X constitute what is called a **Fourier transform pair**.

Remarks on Operator Notation

- For a function x, the Fourier transform of x is denoted using operator notation as $\mathcal{F}x$.
- The Fourier transform of x evaluated at ω is denoted $\mathcal{F}x(\omega)$.
- Note that $\mathcal{F}x$ is a function, whereas $\mathcal{F}x(\omega)$ is a number.
- Similarly, for a function X, the inverse Fourier transform of X is denoted using operator notation as $\mathcal{F}^{-1}X$.
- The inverse Fourier transform of X evaluated at t is denoted $\mathcal{F}^{-1}X(t)$.
- Note that $\mathcal{F}^{-1}X$ is a function, whereas $\mathcal{F}^{-1}X(t)$ is a number.
- With the above said, engineers often abuse notation, and use expressions like those above to mean things different from their proper meanings.
- Since such notational abuse can lead to problems, it is strongly recommended that one refrain from doing this.

Remarks on Dot Notation

- Often, we would like to write an expression for the Fourier transform of a function without explicitly naming the function.
- For example, consider writing an expression for the Fourier transform of the function $v(t) = x(5t-3)$ but without using the name "v".
- It would be incorrect to write "$\mathcal{F}x(5t-3)$" as this is the function $\mathcal{F}x$ evaluated at $5t-3$, which is not the meaning that we wish to convey.
- Also, strictly speaking, it would be incorrect to write "$\mathcal{F}\{x(5t-3)\}$" as the operand of the Fourier transform operator must be a function, and $x(5t-3)$ is a number (i.e., the function x evaluated at $5t-3$).
- Using dot notation, we can write the following strictly-correct expression for the desired Fourier transform: $\mathcal{F}x(5\cdot-3)$.
- In many cases, however, it is probably advisable to avoid employing anonymous (i.e., unnamed) functions, as their use tends to be more error prone in some contexts.

- Since dot notation is less frequently used by engineers, the author has elected to minimize its use herein.
- To avoid ambiguous notation, the following conventions are followed:
    1. in the expression for the operand of a Fourier transform operator, the ***independent variable is assumed to be the variable named "t"*** unless otherwise indicated (i.e., in terms of dot notation, each "t" is treated as if it were a "$\cdot$")
    2. in the expression for the operand of the inverse Fourier transform operator, the ***independent variable is assumed to be the variable named "ω"*** unless otherwise indicated (i.e., in terms of dot notation, each "ω" is treated as if it were a "$\cdot$").
- For example, with these conventions:
    - "$\mathcal{F}\{\cos(t-\tau)\}$" denotes the function that is the Fourier transform of the function $v(t) = \cos(t-\tau)$ (not the Fourier transform of the function $v(\tau) = \cos(t-\tau)$).
    - "$\mathcal{F}^{-1}\{\delta(3\omega-\lambda)\}$" denotes the function that is the inverse Fourier transform of the function $V(\omega) = \delta(3\omega-\lambda)$ (not the inverse Fourier transform of the function $V(\lambda) = \delta(3\omega-\lambda)$).

Section 6.2

Convergence Properties of the Fourier Transform

Convergence of the Fourier Transform

- Consider an arbitrary function x.
- The function x has the Fourier transform representation $\tilde{x}$ given by

$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

- Now, we need to concern ourselves with the convergence properties of this representation.
- In other words, we want to know when $\tilde{x}$ is a valid representation of x.
- Since the Fourier transform is essentially derived from Fourier series, the convergence properties of the Fourier transform are closely related to the convergence properties of Fourier series.

- If a function x is ***continuous*** and ***absolutely integrable*** (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$) and the Fourier transform X of x is absolutely integrable (i.e., $\int_{-\infty}^{\infty} |X(\omega)|\, d\omega < \infty$), then the Fourier transform representation of x converges ***pointwise*** (i.e., $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt\right] e^{j\omega t} d\omega$ for all t).
- Since, in practice, we often encounter functions with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value.

Convergence of the Fourier Transform: Finite-Energy Case

- If a function x is of ***finite energy*** (i.e., $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$), then its Fourier transform representation converges in the ***MSE sense***.
- In other words, if x is of finite energy, then the energy E in the difference function $\tilde{x} - x$ is zero; that is,

$$E = \int_{-\infty}^{\infty} |\tilde{x}(t) - x(t)|^2 \, dt = 0.$$

- Since, in situations of practice interest, the finite-energy condition in the above theorem is often satisfied, the theorem is frequently applicable.
- It is important to note, however, that the condition $E = 0$ does not necessarily imply $\tilde{x}(t) = x(t)$ for all t.
- Thus, the above convergence result does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier transform representation converges.

Convergence of the Fourier Transform: Dirichlet Case

- The **Dirichlet conditions** for the function x are as follows:
  1. The function x is ***absolutely integrable*** (i.e., $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$).
  2. On any finite interval, x has a finite number of maxima and minima (i.e., x is of ***bounded variation***).
  3. On any finite interval, x has a ***finite number of discontinuities*** and each discontinuity is itself ***finite***.
- If a function x satisfies the ***Dirichlet conditions***, then:
  1. The Fourier transform representation $\tilde{x}$ converges pointwise everywhere to x, except at the points of discontinuity of x.
  2. At each point $t = t_a$ of discontinuity, the Fourier transform representation $\tilde{x}$ converges to

$$\tilde{x}(t_a) = \tfrac{1}{2}\left[x(t_a^+) + x(t_a^-)\right],$$

     where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.
- Since most functions tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier transform representation at every point, this result is often very useful in practice.

Section 6.3

Properties of the Fourier Transform

Properties of the (CT) Fourier Transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega)X_2(\omega)$
Time-Domain Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Properties of the (CT) Fourier Transform (Continued)

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega) \rvert^2 \, d\omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}(t/T)$	$\lvert T\rvert \operatorname{sinc}(T\omega/2)$
9	$\frac{\lvert B\rvert}{\pi}\operatorname{sinc} Bt$	$\operatorname{rect}\frac{\omega}{2B}$
10	$e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{\lvert T\rvert}{2}\operatorname{sinc}^2(T\omega/4)$

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$a_1x_1(t) + a_2x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} a_1X_1(\omega) + a_2X_2(\omega),$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Fourier transform.

Time-Domain Shifting (Translation)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

  where t_0 is an arbitrary real constant.

- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

Frequency-Domain Shifting (Modulation)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$e^{j\omega_0 t}x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega - \omega_0),$$
where ω_0 is an arbitrary real constant.
- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

- If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \stackrel{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant.

- This is known as the **dilation (or time/frequency-scaling) property** of the Fourier transform.

Conjugation

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$x^*(t) \overset{\text{CTFT}}{\longleftrightarrow} X^*(-\omega).$$
- This is known as the **conjugation property** of the Fourier transform.

Duality

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega)$$
- This is known as the **duality property** of the Fourier transform.
- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by
$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta) e^{-j\theta\lambda} d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) e^{j\theta\lambda} d\theta.$$
- That is, the forward and inverse Fourier transform equations are identical except for a ***factor of 2π*** and ***different sign*** in the parameter for the exponential function.
- Although the relationship $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform of $x(t)$, the duality property allows us to indirectly infer the Fourier transform of $X(t)$. Consequently, the duality property can be used to effectively ***double*** the number of Fourier transform pairs that we know.

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1 * x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

Time-Domain Multiplication

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} \tfrac{1}{2\pi} X_1 * X_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)d\theta.$$

- This is known as the **(time-domain) multiplication (or frequency-domain convolution) property** of the Fourier transform.
- In other words, multiplication in the time domain becomes convolution in the frequency domain (up to a scale factor of 2π).
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\frac{dx(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\omega X(\omega).$$

- This is known as the **(time-domain) differentiation property** of the Fourier transform.
- Differentiation in the time domain becomes multiplication by $j\omega$ in the frequency domain.
- Of course, by repeated application of the above property, we have that $\left(\frac{d}{dt}\right)^n x(t) \overset{\text{CTFT}}{\longleftrightarrow} (j\omega)^n X(\omega)$.
- The above suggests that the Fourier transform might be a useful tool when working with differential (or integro-differential) equations.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$tx(t) \overset{\text{CTFT}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega).$$

- This is known as the **frequency-domain differentiation property** of the Fourier transform.

- If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

- This is known as the **(time-domain) integration property** of the Fourier transform.
- Whereas differentiation in the time domain corresponds to ***multiplication*** by $j\omega$ in the frequency domain, integration in the time domain is associated with ***division*** by $j\omega$ in the frequency domain.
- Since integration in the time domain becomes division by $j\omega$ in the frequency domain, integration can be easier to handle in the frequency domain.
- The above property suggests that the Fourier transform might be a useful tool when working with integral (or integro-differential) equations.

- Recall that the energy of a function x is given by $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$.
- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

  (i.e., the energy of x and energy of X are equal up to a factor of 2π).
- This relationship is known as **Parseval's relation**.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform ***preserves energy*** (up to a scale factor).

Even/Odd Symmetry

- For a function x with Fourier transform X, the following assertions hold:

$$x \text{ is even} \Leftrightarrow X \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow X \text{ is odd}.$$

- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

- A function x is ***real*** if and only if its Fourier transform X satisfies

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

(i.e., X is ***conjugate symmetric***).

- Thus, for a real-valued function, the portion of the graph of a Fourier transform for negative values of frequency ω is ***redundant***, as it is completely determined by symmetry.
- From properties of complex numbers, one can show that $X(\omega) = X^*(-\omega)$ is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega)$$

(i.e., $|X(\omega)|$ is ***even*** and $\arg X(\omega)$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that X is real.

Fourier Transform of Periodic Functions

- The Fourier transform can be generalized to also handle periodic functions.
- Consider a periodic function x with period T and frequency $\omega_0 = \frac{2\pi}{T}$.
- Define the function x_T as

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases}$$

  (i.e., $x_T(t)$ is equal to $x(t)$ over a single period and zero elsewhere).
- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_T denote the Fourier transforms of x and x_T, respectively.
- The following relationships can be shown to hold:

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0),$$

$$a_k = \frac{1}{T} X_T(k\omega_0), \quad \text{and} \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0).$$

- The Fourier series coefficient sequence a_k is produced by sampling X_T at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$.
- The Fourier transform of a periodic function can only be nonzero at integer multiples of the fundamental frequency.

Section 6.4

Fourier Transform and Frequency Spectra of Functions

Frequency Spectra of Functions

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on functions.
- That is, instead of viewing a function as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a function as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- The Fourier transform of a function x provides a means to ***quantify*** how much information x has at different frequencies.
- The distribution of information in a function over different frequencies is referred to as the ***frequency spectrum*** of the function.

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\omega)$ expressed in ***polar form*** as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j[\omega t + \arg X(\omega)]}d\omega.$$

- In effect, the quantity $|X(\omega)|$ is a ***weight*** that determines how much the complex sinusoid at frequency ω contributes to the integration result x.
- Perhaps, this can be more easily seen if we express the above integral as the ***limit of a sum***, derived from an approximation of the integral using the area of rectangles, as shown on the next slide. [Recall that $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x \to 0} \sum_{k=-\infty}^{\infty} \Delta x f(k\Delta x)$.]

- Expressing the integral (from the previous slide) as the ***limit of a sum***, we obtain

$$x(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega \left| X(\omega') \right| e^{j[\omega' t + \arg X(\omega')]},$$

where $\omega' = k\Delta\omega$.

- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\omega' = k\Delta\omega$ that has had its ***amplitude scaled*** by a factor of $|X(\omega')|$ and has been ***time shifted*** by an amount that depends on $\arg X(\omega')$.
- For a given $\omega' = k\Delta\omega$ (which is associated with the kth term in the summation), the larger $|X(\omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega' t}$ will be, and therefore the larger the contribution the kth term will make to the overall summation.
- In this way, we can use $|X(\omega')|$ as a ***measure*** of how much information a function x has at the frequency ω'.

- The Fourier transform X of the function x is referred to as the **frequency spectrum** of x.
- The magnitude $|X(\omega)|$ of the Fourier transform X is referred to as the **magnitude spectrum** of x.
- The argument $\arg X(\omega)$ of the Fourier transform X is referred to as the **phase spectrum** of x.
- Since the Fourier transform is a function of a real variable, a function can potentially have information at any real frequency.
- Earlier, we saw that for periodic functions, the Fourier transform can only be nonzero at integer multiples of the fundamental frequency.
- So, the Fourier transform and Fourier series give a consistent picture in terms of frequency spectra.
- Since the frequency spectrum is complex (in the general case), it is ***usually represented using two plots***, one showing the magnitude spectrum and one showing the phase spectrum.

- Recall that, for a real function x, the Fourier transform X of x satisfies

$$X(\omega) = X^*(-\omega)$$

(i.e., X is ***conjugate symmetric***), which is equivalent to

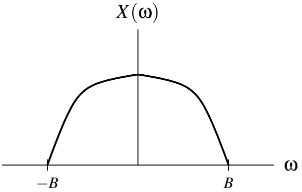
$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega).$$

- Since $|X(\omega)| = |X(-\omega)|$, the magnitude spectrum of a real function is always ***even***.
- Similarly, since $\arg X(\omega) = -\arg X(-\omega)$, the phase spectrum of a real function is always ***odd***.
- Due to the symmetry in the frequency spectra of real functions, we typically ***ignore negative frequencies*** when dealing with such functions.
- In the case of functions that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

- A function x with Fourier transform X is said to be **bandlimited** if, for some nonnegative real constant B, the following condition holds:

$$X(\omega) = 0 \text{ for all } \omega \text{ satisfying } |\omega| > B.$$

- In the context of real functions, we usually refer to B as the **bandwidth** of the function x.
- The (real) function with the Fourier transform X shown below has bandwidth B.



- One can show that a function ***cannot be both time limited and bandlimited***. (This follows from the time/frequency scaling property of the Fourier transform.)

Energy Spectral Density

- By Parseval's relation, the energy E in a function x with Fourier transform X is given by

$$E = \int_{-\infty}^{\infty} E_x(\omega) d\omega,$$

where

$$E_x(\omega) = \tfrac{1}{2\pi} |X(\omega)|^2 .$$

- We refer to E_x as the **energy spectral density** of the function x.
- The function E_x indicates how the energy in x is distributed with respect to frequency.
- For example, the energy contributed by frequencies ω in the range $\omega_1 \leq \omega \leq \omega_2$ is given by

$$\int_{\omega_1}^{\omega_2} E_x(\omega) d\omega.$$

- For a given function x with Fourier transform X, define the function x_T with Fourier transform X_T as

$$x_T(t) = \left(\operatorname{rect} \tfrac{t}{T}\right) x(t).$$

- By Parseval's relation, the power P in the function x is given by

$$P = \int_{-\infty}^{\infty} S_x(\omega) d\omega,$$

where

$$S_x(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} \left| X_T(\omega) \right|^2.$$

- We refer to S_x as the **power spectral density** of the function x.
- The function S_x indicates how the power in x is distributed with respect to frequency.
- For example, the power contributed by frequencies ω in the range $\omega_1 \leq \omega \leq \omega_2$ is given by

$$\int_{\omega_1}^{\omega_2} S_x(\omega) d\omega.$$

Section 6.5

Fourier Transform and LTI Systems

Frequency Response of LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, we have that

$$Y(\omega) = X(\omega)H(\omega).$$

- The function H is called the **frequency response** of the system.
- A LTI system is ***completely characterized*** by its frequency response H.
- The above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output functions.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

Frequency Response of LTI Systems (Continued 1)

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$.
- The quantity $|H(\omega)|$ is called the **magnitude response** of the system.
- The quantity $\arg H(\omega)$ is called the **phase response** of the system.
- Since $Y(\omega) = X(\omega)H(\omega)$, we trivially have that

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)| \quad \text{and} \quad \arg Y(\omega) = \arg X(\omega) + \arg H(\omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.

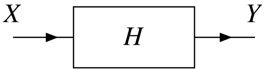
- Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is ***real***, then

$$|H(\omega)| = |H(-\omega)| \quad \text{and} \quad \arg H(\omega) = -\arg H(-\omega)$$

(i.e., the magnitude response $|H(\omega)|$ is ***even*** and the phase response $\arg H(\omega)$ is ***odd***).

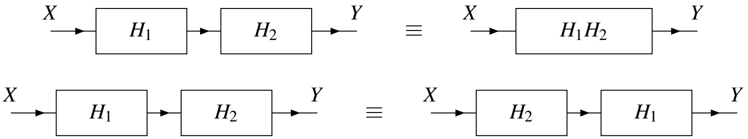
Block Diagram Representations of LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the frequency domain as shown below.

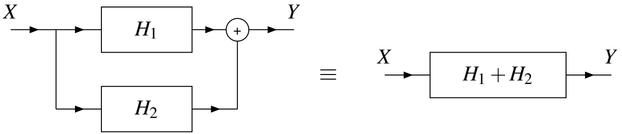


- Since a LTI system is completely characterized by its frequency response, we typically label the system with this quantity.

- The ***series*** interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with frequency response $H_1 H_2$. That is, we have the equivalences shown below.



- The ***parallel*** interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with the frequency response $H_1 + H_2$. That is, we have the equivalence shown below.



- Many LTI systems of practical interest can be represented using an ***Nth-order linear differential equation with constant coefficients***.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t) \quad \text{where} \quad M \leq N.$$

- Let h denote the impulse response of the system, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

- Observe that, for a system of the form considered above, the frequency response is a ***rational function***.

Section 6.6

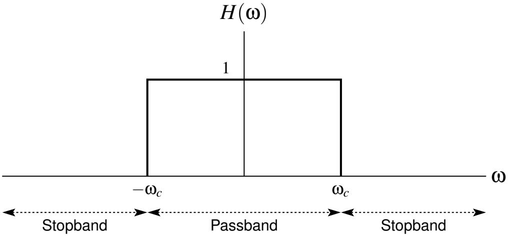
Application: Filtering

- In many applications, we want to ***modify the spectrum*** of a function by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a function is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

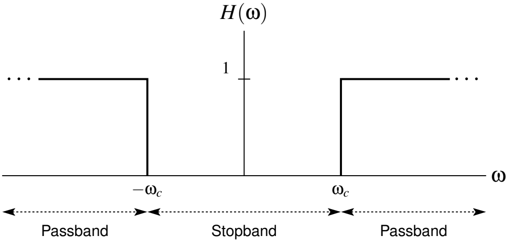


Ideal Highpass Filter

- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

  where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

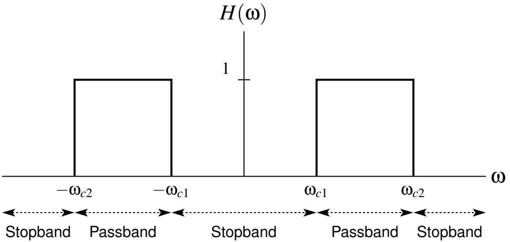


Ideal Bandpass Filter

- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.



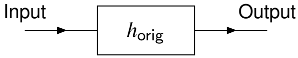
Section 6.7

Application: Equalization

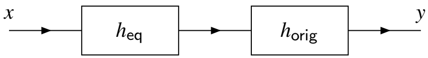
Equalization

- Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand.
- As a result, we would like to change the frequency response of the system to be something more desirable.
- This process of modifying the frequency response in this way is referred to as **equalization**. [Essentially, equalization is just a filtering operation.]
- Equalization is used in many applications.
- In real-world ***communication systems***, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel.
- In ***audio applications***, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, equalization can be used to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

Equalization (Continued)



Original System



New System with Equalization

- Let H_{orig} denote the frequency response of ***original*** system (i.e., without equalization).
- Let H_{d} denote the ***desired*** frequency response.
- Let H_{eq} denote the frequency response of the ***equalizer***.
- The new system with equalization has frequency response

$$H_{\text{new}}(\omega) = H_{\text{eq}}(\omega) H_{\text{orig}}(\omega).$$

- By choosing $H_{\text{eq}}(\omega) = H_{\text{d}}(\omega)/H_{\text{orig}}(\omega)$, the new system with equalization will have the frequency response

$$H_{\text{new}}(\omega) = \left[H_{\text{d}}(\omega)/H_{\text{orig}}(\omega)\right] H_{\text{orig}}(\omega) = H_{\text{d}}(\omega).$$

- In effect, by using an equalizer, we can obtain a new system with the frequency response that we desire.

Section 6.8

Application: Circuit Analysis

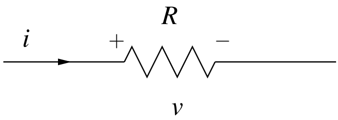
- A **resistor** is a circuit element that opposes the flow of electric current.
- A resistor with resistance R is governed by the relationship

$$v(t) = Ri(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

where v and i respectively denote the voltage across and current through the resistor as a function of time.
- In the frequency domain, the above relationship becomes

$$V(\omega) = RI(\omega) \quad \left(\text{or equivalently, } I(\omega) = \tfrac{1}{R}V(\omega)\right),$$

where V and I denote the Fourier transforms of v and i, respectively.
- In circuit diagrams, a resistor is denoted by the symbol shown below.



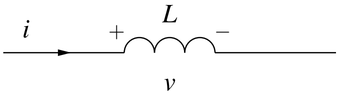
- An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor with inductance L is governed by the relationship

$$v(t) = L\frac{d}{dt}i(t) \quad \left(\text{or equivalently, } i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau\right),$$

  where v and i respectively denote the voltage across and current through the inductor as a function of time.
- In the frequency domain, the above relationship becomes

$$V(\omega) = j\omega L I(\omega) \quad \left(\text{or equivalently, } I(\omega) = \frac{1}{j\omega L}V(\omega)\right),$$

  where V and I denote the Fourier transforms of v and i, respectively.
- In circuit diagrams, an inductor is denoted by the symbol shown below.



Capacitors

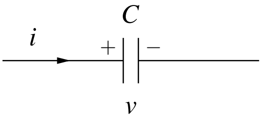
- A **capacitor** is a circuit element that stores electric charge.
- A capacitor with capacitance C is governed by the relationship

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \left(\text{or equivalently, } i(t) = C\frac{d}{dt}v(t)\right),$$

  where v and i respectively denote the voltage across and current through the capacitor as a function of time.
- In the frequency domain, the above relationship becomes

$$V(\omega) = \frac{1}{j\omega C}I(\omega) \quad \left(\text{or equivalently, } I(\omega) = j\omega CV(\omega)\right),$$

  where V and I denote the Fourier transforms of v and i, respectively.
- In circuit diagrams, a capacitor is denoted by the symbol shown below.



- The Fourier transform is a very useful tool for circuit analysis.
- The utility of the Fourier transform is partly due to the fact that the ***differential/integral*** equations that describe inductors and capacitors are much simpler to express in the Fourier domain than in the time domain.

Section 6.9

Application: Amplitude Modulation (AM)

Motivation for Amplitude Modulation (AM)

- In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal.
- For example, voice/audio signals typically have information in the range of 0 to 22 kHz.
- Often, it is not practical to transmit such a signal using its original frequency range.
- Two potential problems with such an approach are:
  1. interference; and
  2. constraints on antenna length.
- Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference.
- Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals.
- For the preceding reasons, we often need to change the frequency range associated with a signal before transmission.

Transmitter

Receiver

- The transmitter is characterized by

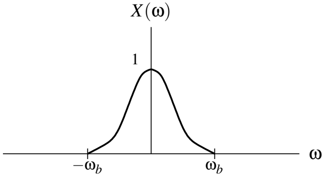
$$y(t) = e^{j\omega_c t}x(t) \iff Y(\omega) = X(\omega - \omega_c).$$

- The receiver is characterized by

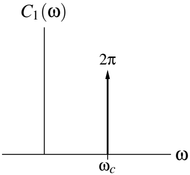
$$\hat{x}(t) = e^{-j\omega_c t}y(t) \iff \hat{X}(\omega) = Y(\omega + \omega_c).$$

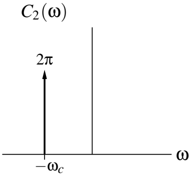
- Clearly, $\hat{x}(t) = e^{j\omega_c t}e^{-j\omega_c t}x(t) = x(t)$.

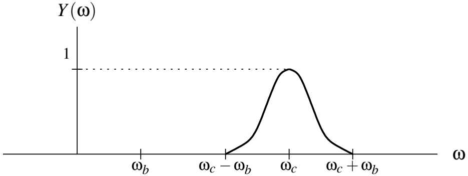
Trivial Amplitude Modulation (AM) System: Example



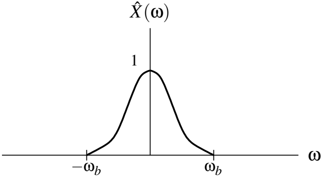
Transmitter Input



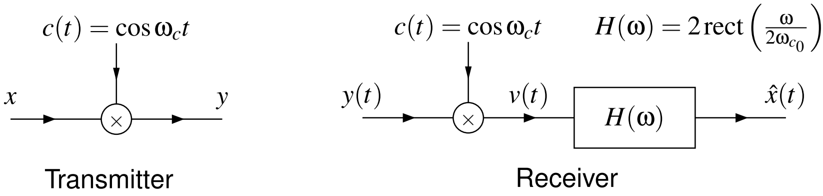




Transmitter Output



Receiver Output



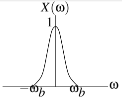
- Suppose that $X(\omega) = 0$ for all $\omega \notin [-\omega_b, \omega_b]$.
- The transmitter is characterized by

$$Y(\omega) = \tfrac{1}{2}\left[X(\omega+\omega_c) + X(\omega-\omega_c)\right].$$

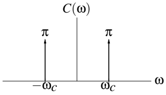
- The receiver is characterized by

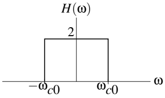
$$\hat{X}(\omega) = \left[Y(\omega+\omega_c) + Y(\omega-\omega_c)\right] \mathrm{rect}\left(\frac{\omega}{2\omega_{c_0}}\right).$$

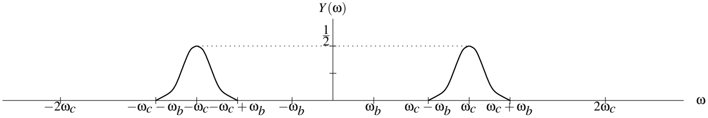
- If $\omega_b < \omega_{c_0} < 2\omega_c - \omega_b$, we have $\hat{X}(\omega) = X(\omega)$ (implying $\hat{x}(t) = x(t)$).



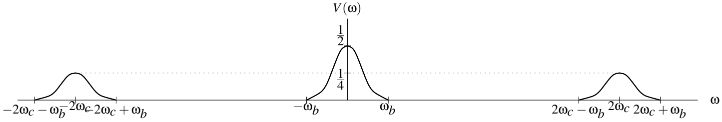
Transmitter Input

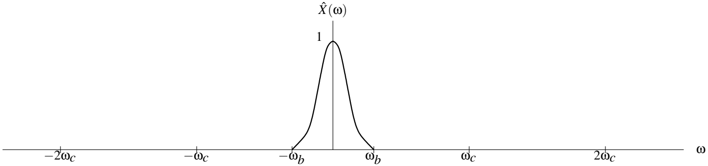






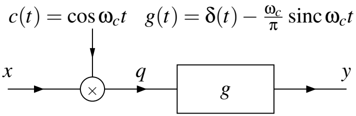
Transmitter Output



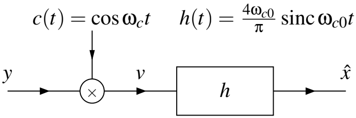


Receiver Output

Single-Sideband Suppressed-Carrier (SSB-SC) AM



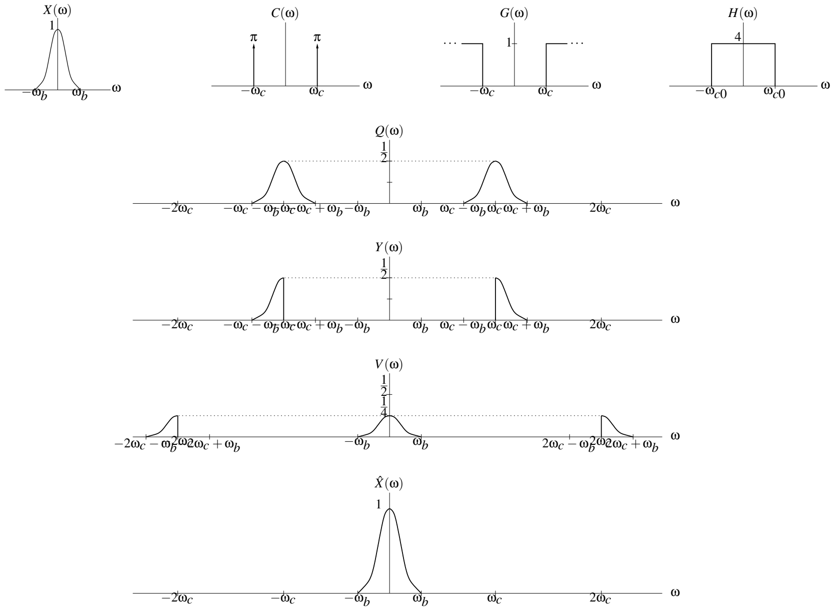
Transmitter



Receiver

- The basic analysis of the SSB-SC AM system is similar to the DSB-SC AM system.
- SSB-SC AM requires half as much bandwidth for the transmitted signal as DSB-SC AM.

SSB-SC AM: Example

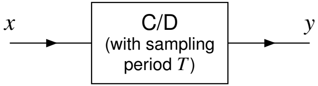


Section 6.10

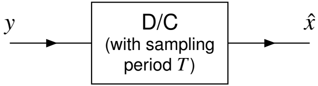
Application: Sampling and Interpolation

Sampling and Interpolation

- Often, we want to be able to ***convert*** between continuous-time and discrete-time representations of a signal.
- This is accomplished through processes known as ***sampling*** and ***interpolation***.
- The ***sampling*** process, which is performed by an **ideal continuous-time to discrete-time (C/D) converter** shown below, transforms a continuous-time signal x to a discrete-time signal y.



- The ***interpolation*** process, which is performed by an **ideal discrete-time to continuous-time (D/C) converter** shown below, transforms a discrete-time signal y to a continuous-time signal $\hat{x}$.



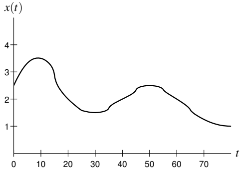
- Note that, unless very special conditions are met, the sampling process loses information (i.e., is ***not invertible***).

Periodic Sampling

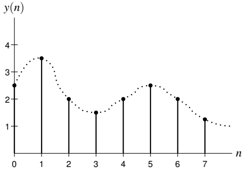
- Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**.
- With this scheme, a sequence y of samples is obtained from a continuous-time signal x according to the relation

$$y(n) = x(Tn) \quad \text{for all integer } n,$$

  where T is a positive real constant.
- As a matter of terminology, we refer to T as the **sampling period**, and $\omega_s = 2\pi/T$ as the (angular) **sampling frequency**.
- An example of periodic sampling is shown below, where the original continuous-time signal x has been sampled with ***sampling period*** $T = 10$, yielding the sequence y.



Original Signal



Sampled Signal

Periodic Sampling (Continued)

- The sampling process is not generally invertible.
- In the absence of any constraints, a continuous-time signal cannot usually be uniquely determined from a sequence of its equally-spaced samples.
- Consider, for example, the continuous-time signals x_1 and x_2 given by

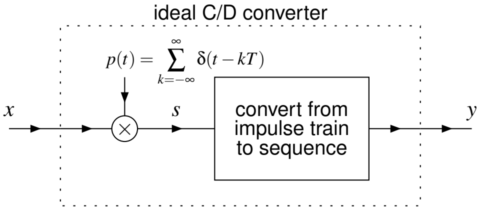
$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

- If we sample each of these signals with the sampling period $T = 1$, we obtain the respective sequences

$$y_1(n) = x_1(Tn) = x_1(n) = 0 \quad \text{and}$$
$$y_2(n) = x_2(Tn) = \sin(2\pi n) = 0.$$

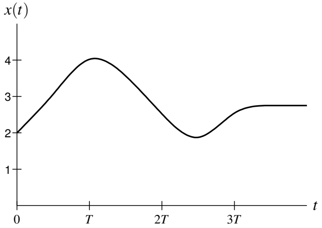
- Thus, $y_1(n) = y_2(n)$ for all n, although $x_1(t) \neq x_2(t)$ for all noninteger t.
- Fortunately, under certain circumstances, a continuous-time signal can be recovered exactly from its samples.

- An **impulse train** is a function of the form $v(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - kT)$, where a_k and T are real constants (i.e., $v(t)$ consists of weighted impulses spaced apart by T).
- For the purposes of analysis, sampling with sampling period T and frequency $\omega_s = \frac{2\pi}{T}$ can be modelled as shown below.

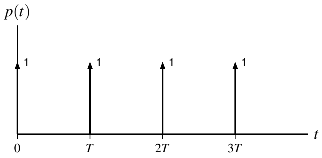


- The sampling of a continuous-time signal x to produce a sequence y consists of the following two steps (in order):
  1. Multiply the signal x to be sampled by a periodic impulse train p, yielding the impulse train s.
  2. Convert the impulse train s to a sequence y, by forming a sequence from the weights of successive impulses in s.

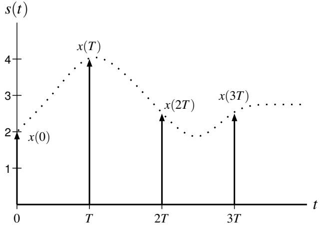
Model of Sampling: Various Signals



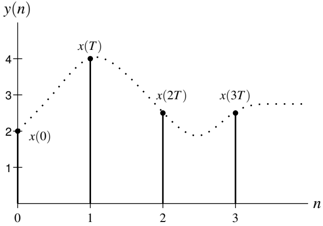
Input Signal (Continuous-Time)



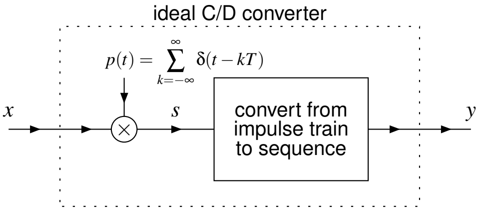
Periodic Impulse Train



Impulse-Sampled Signal *(Continuous-Time)*



Output Sequence *(Discrete-Time)*



- In the time domain, the impulse-sampled signal s is given by

$$s(t) = x(t)p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

- In the Fourier domain, the preceding equation becomes

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

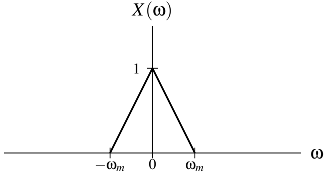
- Thus, the spectrum of the impulse-sampled signal s is a scaled sum of an infinite number of ***shifted copies*** of the spectrum of the original signal x.

Model of Sampling: Aliasing

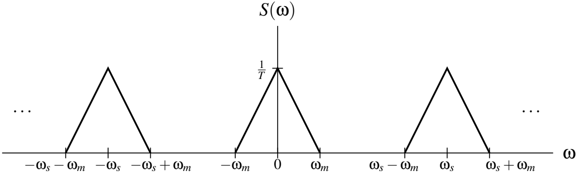
- Consider frequency spectrum S of the impulse-sampled signal s given by

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

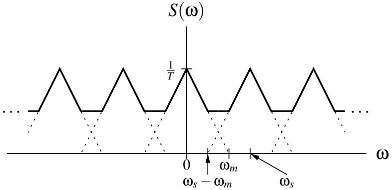
- The function S is a scaled sum of an infinite number of ***shifted copies*** of X.
- Two distinct behaviors can result in this summation, depending on ω_s and the bandwidth of x.
- In particular, the nonzero portions of the different shifted copies of X can either:
  1. overlap; or
  2. not overlap.
- In the case where overlap occurs, the various shifted copies of X add together in such a way that the original shape of X is lost. This phenomenon is known as **aliasing**.
- When aliasing occurs, the original signal x cannot be recovered from its samples in y.



Spectrum of Input Signal
(Bandwidth ω_m)



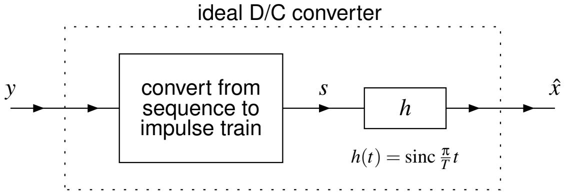
Spectrum of Impulse-Sampled Signal:
No Aliasing Case
($\omega_s > 2\omega_m$)



Spectrum of Impulse-Sampled Signal:
Aliasing Case
($\omega_s \leq 2\omega_m$)

Model of Interpolation

- For the purposes of analysis, interpolation can be modelled as shown below.



- The inverse Fourier transform h of H is $h(t) = \operatorname{sinc}(\pi t / T)$.
- The reconstruction of a continuous-time signal x from its sequence y of samples (i.e., bandlimited interpolation) consists of the following two steps (in order):
  1. Convert the sequence y to the impulse train s, by using the elements in the sequence as the weights of successive impulses in the impulse train.
  2. Apply a lowpass filter to s to produce $\hat{x}$.
- The lowpass filter is used to eliminate the extra copies of the original signal's spectrum present in the spectrum of the impulse-sampled signal s.

- In more detail, the reconstruction process proceeds as follows.
- First, we convert the sequence y to the impulse train s to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - Tn).$$

- Then, we filter the resulting signal s with the lowpass filter having impulse response h, yielding

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left[\tfrac{\pi}{T}(t - Tn)\right].$$

- **Sampling Theorem.** Let x be a function with Fourier transform X, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., x is bandlimited to frequencies $[-\omega_M, \omega_M]$). Then, x is uniquely determined by its samples $y(n) = x(Tn)$ for all integer n, if

$$\omega_s > 2\omega_M,$$

where $\omega_s = 2\pi/T$. The preceding inequality is known as the **Nyquist condition**. If this condition is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}(\tfrac{\pi}{T}(t - Tn)),$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}(\tfrac{\omega_s}{2} t - \pi n).$$

- We call $\omega_s/2$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**.

Part 7

Laplace Transform (LT)

Motivation Behind the Laplace Transform

- Another important mathematical tool in the study of signals and systems is known as the Laplace transform.
- The Laplace transform can be viewed as a ***generalization of the Fourier transform***.
- Due to its more general nature, the Laplace transform has a number of ***advantages*** over the Fourier transform.
- First, the Laplace transform representation ***exists for some functions that do not have a Fourier transform representation***. So, we can handle some functions with the Laplace transform that cannot be handled with the Fourier transform.
- Second, since the Laplace transform is a more general tool, it can provide ***additional insights*** beyond those facilitated by the Fourier transform.

- Earlier, we saw that complex exponentials are eigenfunctions of LTI systems.
- In particular, for a LTI system $\mathcal{H}$ with impulse response h, we have that

$$\mathcal{H}\{e^{st}\}(t) = H(s)e^{st} \quad \text{where} \quad H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

- Previously, we referred to H as the system function.
- As it turns out, H is the Laplace transform of h.
- Since the Laplace transform has already appeared earlier in the context of LTI systems, it is clearly a useful tool.
- Furthermore, as we will see, the Laplace transform has many additional uses.

Section 7.1

Laplace Transform

- The (bilateral) **Laplace transform** of the function x, denoted $\mathcal{L}x$ or X, is defined as

$$\mathcal{L}x(s) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

- The **inverse Laplace transform** of X, denoted $\mathcal{L}^{-1}X$ or x, is then given by

$$\mathcal{L}^{-1}X(t) = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds,$$

  where $\mathrm{Re}(s) = \sigma$ is in the ROC of X. (Note that this is a ***contour integration***, since s is complex.)

- We refer to x and X as a **Laplace transform pair** and denote this relationship as

$$x(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s).$$

- In practice, we do not usually compute the inverse Laplace transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

Bilateral and Unilateral Laplace Transforms

- Two different versions of the Laplace transform are commonly used:
  1. the ***bilateral*** (or ***two-sided***) Laplace transform; and
  2. the ***unilateral*** (or ***one-sided***) Laplace transform.
- The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions.
- As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the ***lower limit of integration***.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0.
- For the most part, we will focus our attention primarily on the bilateral Laplace transform.
- We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations.
- Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean ***bilateral*** Laplace transform.

Remarks on Operator Notation

- For a function x, the Laplace transform of x is denoted using operator notation as $\mathcal{L}x$.
- The Laplace transform of x evaluated at s is denoted $\mathcal{L}x(s)$.
- Note that $\mathcal{L}x$ is a function, whereas $\mathcal{L}x(s)$ is a number.
- Similarly, for a function X, the inverse Laplace transform of X is denoted using operator notation as $\mathcal{L}^{-1}X$.
- The inverse Laplace transform of X evaluated at t is denoted $\mathcal{L}^{-1}X(t)$.
- Note that $\mathcal{L}^{-1}X$ is a function, whereas $\mathcal{L}^{-1}X(t)$ is a number.
- With the above said, engineers often abuse notation, and use expressions like those above to mean things different from their proper meanings.
- Since such notational abuse can lead to problems, it is strongly recommended that one refrain from doing this.

- Often, we would like to write an expression for the Laplace transform of a function without explicitly naming the function.
- For example, consider writing an expression for the Laplace transform of the function $v(t) = x(5t-3)$ but without using the name "v".
- It would be incorrect to write "$\mathcal{L}x(5t-3)$" as this is the function $\mathcal{L}x$ evaluated at $5t-3$, which is not the meaning that we wish to convey.
- Also, strictly speaking, it would be incorrect to write "$\mathcal{L}\{x(5t-3)\}$" as the operand of the Laplace transform operator must be a function, and $x(5t-3)$ is a number (i.e., the function x evaluated at $5t-3$).
- Using dot notation, we can write the following strictly-correct expression for the desired Laplace transform: $\mathcal{L}\{x(5\cdot-3)\}$.
- In many cases, however, it is probably advisable to avoid employing anonymous (i.e., unnamed) functions, as their use tends to be more error prone in some contexts.

Remarks on Notational Conventions

- Since dot notation is less frequently used by engineers, the author has elected to minimize its use herein.
- To avoid ambiguous notation, the following conventions are followed:
  1. in the expression for the operand of a Laplace transform operator, the ***independent variable is assumed to be the variable named "t"*** unless otherwise indicated (i.e., in terms of dot notation, each "t" is treated as if it were a "$\cdot$")
  2. in the expression for the operand of the inverse Laplace transform operator, the ***independent variable is assumed to be the variable named "s"*** unless otherwise indicated (i.e., in terms of dot notation, each "s" is treated as if it were a "$\cdot$").
- For example, with these conventions:
  - "$\mathcal{L}\{(t-\tau)u(t-\tau)\}$" denotes the function that is the Laplace transform of the function $v(t) = (t-\tau)u(t-\tau)$ (not the Laplace transform of the function $v(\tau) = (t-\tau)u(t-\tau)$).
  - "$\mathcal{L}^{-1}\{\frac{1}{s^2-\lambda}\}$" denotes the function that is the inverse Laplace transform of the function $V(s) = \{\frac{1}{s^2-\lambda}\}$ (not the inverse Laplace transform of the function $V(\lambda) = \{\frac{1}{s^2-\lambda}\}$).

Relationship Between Laplace and Fourier Transforms

- Let X and X_F denote the Laplace and (CT) Fourier transforms of x, respectively.
- The function X evaluated at $j\omega$ (where ω is real) yields $X_\mathsf{F}(\omega)$. That is,

$$X(j\omega) = X_\mathsf{F}(\omega).$$

- Due to the preceding relationship, the Fourier transform of x is sometimes written as $X(j\omega)$.
- The function X evaluated at an arbitrary complex value $s = \sigma + j\omega$ (where $\sigma = \mathrm{Re}(s)$ and $\omega = \mathrm{Im}(s)$) can also be expressed in terms of a Fourier transform involving x. In particular, we have

$$X(\sigma + j\omega) = X'_\mathsf{F}(\omega),$$

  where X'_F is the (CT) Fourier transform of $x'(t) = e^{-\sigma t}x(t)$.
- So, in general, the Laplace transform of x is the Fourier transform of an exponentially-weighted version of x.
- Due to this weighting, the Laplace transform of a function may exist when the Fourier transform of the same function does not.

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Section 7.2

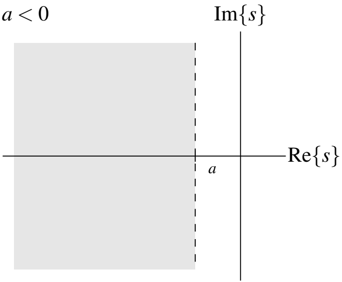
Region of Convergence (ROC)

- The set R of all complex numbers s satisfying

$$\mathrm{Re}(s) < a$$

for some real constant a is said to be a **left-half plane (LHP)**.

- Some examples of LHPs are shown below.



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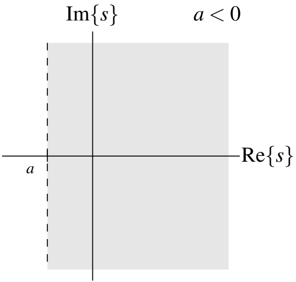
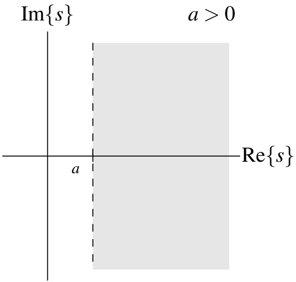
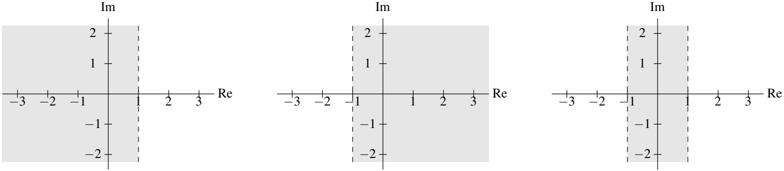




- For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B.
- An illustrative example of set intersection is shown below.



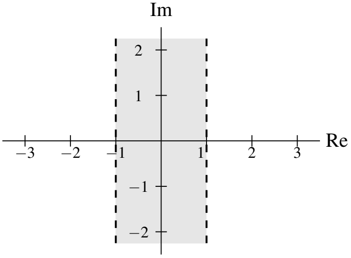
R_1 $\qquad$ R_2 $\qquad$ $R_1 \cap R_2$

- For a set S and a scalar constant a, $S+a$ denotes the set given by

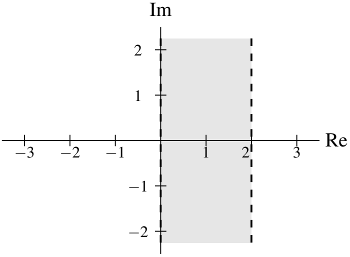
$$S+a=\{z+a : z \in S\}$$

(i.e., $S+a$ is the set formed by adding a to each element of S).

- An illustrative example is given below.



R



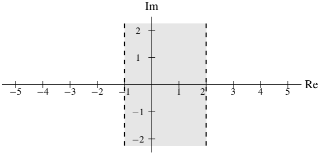
$R+1$

- For a set S and a scalar constant a, aS denotes the set given by

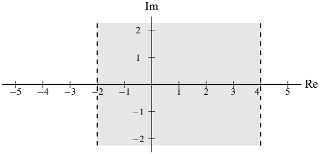
$$aS = \{az : z \in S\}$$

(i..e, aS is the set formed by multiplying each element of S by a).

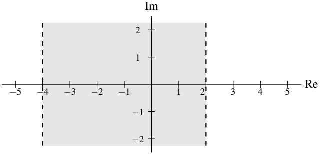
- An illustrative example is given below.



R



$2R$



$-2R$

Region of Convergence (ROC)

- As we saw earlier, for a function x, the complete specification of its Laplace transform X requires not only an algebraic expression for X, but also the ROC associated with X.
- Two very different functions can have the same algebraic expressions for X.
- Now, we examine some of the constraints on the ROC (of the Laplace transform) for various classes of functions.

Properties of the ROC

1. The ROC of the Laplace transform X consists of ***strips parallel to the imaginary axis*** in the complex plane.
2. If the Laplace transform X is a ***rational*** function, the ROC ***does not contain any poles***, and the ROC is ***bounded by poles or extends to infinity***.
3. If the function x is ***finite duration*** and its Laplace transform $X(s)$ converges for some value of s, then $X(s)$ converges for ***all values*** of s (i.e., the ROC is the entire complex plane).
4. If the function x is ***right sided*** and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then all values of s for which $\mathrm{Re}(s) > \sigma_0$ must also be in the ROC (i.e., the ROC is a ***RHP*** including $\mathrm{Re}(s) = \sigma_0$).
5. If the function x is ***left sided*** and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then all values of s for which $\mathrm{Re}(s) < \sigma_0$ must also be in the ROC (i.e., the ROC is a ***LHP*** including $\mathrm{Re}(s) = \sigma_0$).

Properties of the ROC (Continued)

6. If the function x is ***two sided*** and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then the ROC will consist of a ***strip*** in the complex plane that includes the line $\mathrm{Re}(s) = \sigma_0$.
7. If the Laplace transform X of the function x is ***rational*** (with at least one pole), then:
    1. If x is ***right sided***, the ROC of X is to the right of the rightmost pole of X (i.e., the ***RHP*** to the ***right of the rightmost pole***).
    2. If x is ***left sided***, the ROC of X is to the left of the leftmost pole of X (i.e., the ***LHP*** to the ***left of the leftmost pole***).

- Some of the preceding properties are ***redundant*** (e.g., properties 1, 2, 4, and 5 imply property 7).
- Since every function can be classified as one of finite duration, left sided but not right sided, right sided but not left sided, or two sided, we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of a LHP, RHP, vertical strip, the entire complex plane, or the empty set. Thus, the ROC must be a ***connected region***.

Section 7.3

Properties of the Laplace Transform

Properties of the Laplace Transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\operatorname{Re}(s_0)$
Time/Laplace-Domain Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\operatorname{Re}(s)>0\}$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Laplace Transform Pairs

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\operatorname{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\operatorname{Re}(s) < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\operatorname{Re}(s) > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\operatorname{Re}(s) < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\operatorname{Re}(s) > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\operatorname{Re}(s) < -a$
8	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\operatorname{Re}(s) > -a$
9	$-t^n e^{-at}u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\operatorname{Re}(s) < -a$
10	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\operatorname{Re}(s) > 0$
11	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\operatorname{Re}(s) > 0$
12	$[e^{-at}\cos\omega_0 t]u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}(s) > -a$
13	$[e^{-at}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}(s) > -a$

- If $x_1(t) \xleftrightarrow{\text{LT}} X_1(s)$ with ROC R_1 and $x_2(t) \xleftrightarrow{\text{LT}} X_2(s)$ with ROC R_2, then

  $$a_1x_1(t) + a_2x_2(t) \xleftrightarrow{\text{LT}} a_1X_1(s) + a_2X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2,$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Laplace transform.
- The ROC always contains the intersection but could be larger (in the case that pole-zero cancellation occurs).

Time-Domain Shifting

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

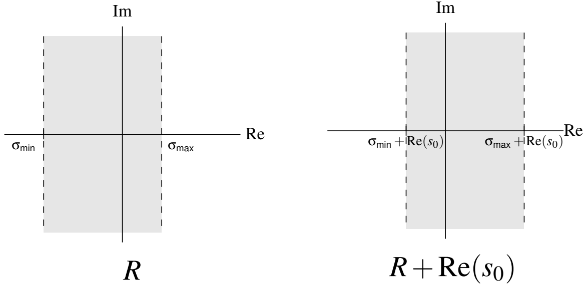
$$x(t-t_0) \overset{\text{LT}}{\longleftrightarrow} e^{-st_0}X(s) \text{ with ROC } R,$$

  where t_0 is an arbitrary real constant.
- This is known as the **time-domain shifting property** of the Laplace transform.

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$e^{s_0 t}x(t) \overset{\text{LT}}{\longleftrightarrow} X(s-s_0) \text{ with ROC } R+\operatorname{Re}(s_0),$$

  where s_0 is an arbitrary complex constant.
- This is known as the **Laplace-domain shifting property** of the Laplace transform.
- As illustrated below, the ROC R is ***shifted*** right by $\operatorname{Re}(s_0)$.

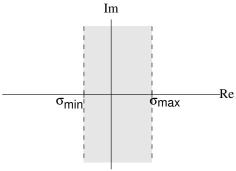


- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

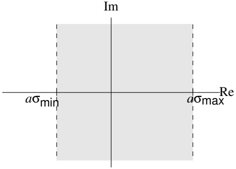
$$x(at) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{with ROC } R_1 = aR,$$

where a is a nonzero real constant.

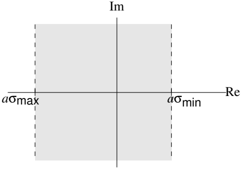
- This is known as the **(time-domain/Laplace-domain) scaling property** of the Laplace transform.
- As illustrated below, the ROC R is ***scaled*** and ***possibly flipped*** left to right.



R



aR, $a > 0$



aR, $a < 0$

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x^*(t) \overset{\text{LT}}{\longleftrightarrow} X^*(s^*) \quad \text{with ROC } R.$$

- This is known as the **conjugation property** of the Laplace transform.

- If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

$$x_1 * x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)X_2(s) \text{ with ROC containing } R_1 \cap R_2.$$

- This is known as the **time-domain convolution property** of the Laplace transform.
- The ROC always contains the intersection but can be larger than the intersection (if pole-zero cancellation occurs).
- Convolution in the time domain becomes ***multiplication*** in the Laplace domain.
- Consequently, it is often much easier to work with LTI systems in the Laplace domain, rather than the time domain.

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\frac{dx(t)}{dt} \overset{\text{LT}}{\longleftrightarrow} sX(s) \text{ with ROC containing } R.$$

- This is known as the **time-domain differentiation property** of the Laplace transform.
- The ROC always contains R but can be larger than R (if pole-zero cancellation occurs).
- Differentiation in the time domain becomes ***multiplication by s*** in the Laplace domain.
- Consequently, it can often be much easier to work with differential equations in the Laplace domain, rather than the time domain.

Laplace-Domain Differentiation

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$-tx(t) \overset{\text{LT}}{\longleftrightarrow} \frac{dX(s)}{ds} \text{ with ROC } R.$$

- This is known as the **Laplace-domain differentiation property** of the Laplace transform.

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s}X(s) \text{ with ROC containing } R \cap \{\operatorname{Re}(s) > 0\}.$$

- This is known as the **time-domain integration property** of the Laplace transform.
- The ROC always contains at least $R \cap \{\operatorname{Re}(s) > 0\}$ but can be larger (if pole-zero cancellation occurs).
- Integration in the time domain becomes ***division by s*** in the Laplace domain.
- Consequently, it is often much easier to work with integral equations in the Laplace domain, rather than the time domain.

Initial Value Theorem

- For a function x with Laplace transform X, if x is ***causal*** and contains ***no impulses or higher order singularities at the origin***, then

$$x(0^+) = \lim_{s\to\infty} sX(s),$$

  where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t.
- This result is known as the **initial value theorem**.

Final Value Theorem

- For a function x with Laplace transform X, if x is ***causal*** and $x(t)$ has a ***finite limit*** as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

- This result is known as the **final value theorem**.
- Sometimes the initial and final value theorems are useful for checking for errors in Laplace transform calculations. For example, if we had made a mistake in computing $X(s)$, the values obtained from the initial and final value theorems would most likely disagree with the values obtained directly from the original expression for $x(t)$.

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Section 7.4

Determination of Inverse Laplace Transform

Finding Inverse Laplace Transform

- Recall that the inverse Laplace transform x of X is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds,$$

where $\mathrm{Re}(s) = \sigma$ is in the ROC of X.
- Unfortunately, the above contour integration can often be ***quite tedious*** to compute.
- Consequently, we do not usually compute the inverse Laplace transform directly using the above equation.
- For rational functions, the inverse Laplace transform can be more easily computed using ***partial fraction expansions***.
- Using a partial fraction expansion, we can express a rational function as a sum of lower-order rational functions whose inverse Laplace transforms can typically be found in tables.

Section 7.5

Laplace Transform and LTI Systems

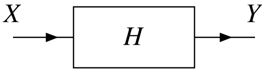
System Function of LTI Systems

- Consider a LTI system with input x, output y, and impulse response h. Let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, the system is characterized in the Laplace domain by

$$Y(s) = X(s)H(s).$$

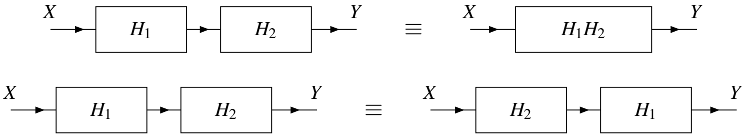
- As a matter of terminology, we refer to H as the **system function** (or **transfer function**) of the system (i.e., the system function is the Laplace transform of the impulse response).
- When viewed in the Laplace domain, a LTI system forms its output by multiplying its input with its system function.
- A LTI system is ***completely characterized*** by its system function H.
- If the ROC of H includes the imaginary axis, then $H(j\omega)$ is the ***frequency response*** of the LTI system.

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the Laplace domain as shown below.

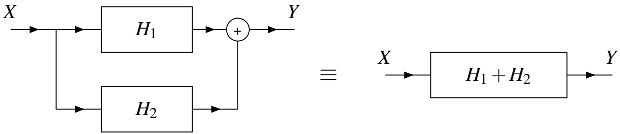


- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

- The ***series*** interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with system function $H_1 H_2$. That is, we have the equivalences shown below.



- The ***parallel*** interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with the system function $H_1 + H_2$. That is, we have the equivalence shown below.



- If a LTI system is ***causal***, its impulse response is causal, and therefore ***right sided***. From this, we have the result below.
- **Theorem.** The ROC associated with the system function of a ***causal*** LTI system is a ***right-half plane*** or the entire complex plane.
- In general, the ***converse*** of the above theorem is ***not necessarily true***. That is, if the ROC of the system function is a RHP or the entire complex plane, it is not necessarily true that the system is causal.
- If the system function is ***rational***, however, we have that the converse does hold, as indicated by the theorem below.
- **Theorem.** For a LTI system with a ***rational*** system function H, ***causality*** of the system is ***equivalent*** to the ROC of H being the ***right-half plane*** to the right of the rightmost pole or, if H has no poles, the entire complex plane.

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is ***BIBO stable*** if and only if the ROC of its system function H contains the ***imaginary axis*** (i.e., $\operatorname{Re}(s) = 0$).
- **Theorem.** A ***causal*** LTI system with a (proper) ***rational*** system function H is BIBO stable if and only if all of the poles of H lie in the left half of the plane (i.e., all of the poles have ***negative real parts***).

Invertibility

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(s)H_{\text{inv}}(s) = 1,$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(s) = \frac{1}{H(s)}.$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is ***not necessarily unique***.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in ***one specific choice*** of inverse system (due to these additional constraints of stability and/or causality).

LTI Systems and Differential Equations

- Many LTI systems of practical interest can be represented using an ***Nth-order linear differential equation with constant coefficients***.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t) \quad \text{where} \quad M \leq N.$$

- Let h denote the impulse response of the system, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

- Observe that, for a system of the form considered above, the system function is always ***rational***.

Section 7.6

Application: Circuit Analysis

Resistors

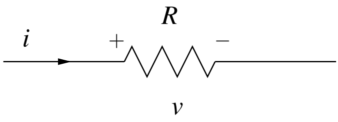
- A **resistor** is a circuit element that opposes the flow of electric current.
- A resistor with resistance R is governed by the relationship

$$v(t) = Ri(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

  where v and i respectively denote the voltage across and current through the resistor as a function of time.
- In the Laplace domain, the above relationship becomes

$$V(s) = RI(s) \quad \left(\text{or equivalently, } I(s) = \tfrac{1}{R}V(s)\right),$$

  where V and I denote the Laplace transforms of v and i, respectively.
- In circuit diagrams, a resistor is denoted by the symbol shown below.



Inductors

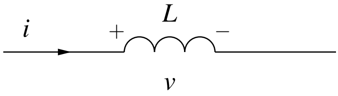
- An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor with inductance L is governed by the relationship

$$v(t) = L\frac{d}{dt}i(t) \quad \left(\text{or equivalently, } i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau\right),$$

  where v and i respectively denote the voltage across and current through the inductor as a function of time.
- In the Laplace domain, the above relationship becomes

$$V(s) = sLI(s) \quad \left(\text{or equivalently, } I(s) = \frac{1}{sL}V(s)\right),$$

  where V and I denote the Laplace transforms of v and i, respectively.
- In circuit diagrams, an inductor is denoted by the symbol shown below.



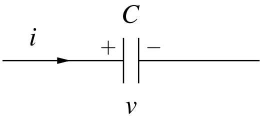
- A **capacitor** is a circuit element that stores electric charge.
- A capacitor with capacitance C is governed by the relationship

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \left(\text{or equivalently, } i(t) = C\frac{d}{dt}v(t)\right),$$

  where v and i respectively denote the voltage across and current through the capacitor as a function of time.
- In the Laplace domain, the above relationship becomes

$$V(s) = \frac{1}{sC}I(s) \quad \left(\text{or equivalently, } I(s) = sCV(s)\right),$$

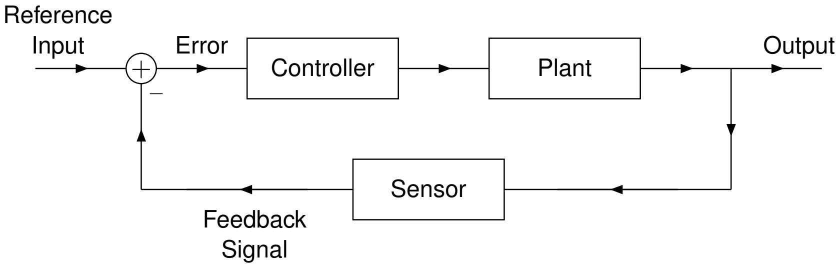
  where V and I denote the Laplace transforms of v and i, respectively.
- In circuit diagrams, a capacitor is denoted by the symbol shown below.



- The Laplace transform is a very useful tool for circuit analysis.
- The utility of the Laplace transform is partly due to the fact that the ***differential/integral*** equations that describe inductors and capacitors are much simpler to express in the Laplace domain than in the time domain.

Section 7.7

Application: Analysis of Control Systems



- **input**: ***desired value*** of the quantity to be controlled
- **output**: ***actual value*** of the quantity to be controlled
- **error**: ***difference*** between the desired and actual values
- **plant**: system to be controlled
- **sensor**: device used to measure the actual output
- **controller**: device that monitors the error and changes the input of the plant with the goal of forcing the error to zero

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the Laplace domain than in the time domain.
- Therefore, the Laplace domain is extremely useful for the stability analysis of systems.

Section 7.8

Unilateral Laplace Transform

Unilateral Laplace Transform

- The **unilateral Laplace transform** of the function x, denoted $\mathcal{L}_{\mathrm{u}}\{x\}$ or X, is defined as

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt.$$

- The unilateral Laplace transform is related to the bilateral Laplace transform as follows:

$$\mathcal{L}_{\mathrm{u}}\{x\}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt = \mathcal{L}\{xu\}(s).$$

- In other words, the unilateral Laplace transform of the function x is simply the bilateral Laplace transform of the function xu.
- Since $\mathcal{L}_{\mathrm{u}}\{x\} = \mathcal{L}\{xu\}$ and xu is always a ***right-sided*** function, the ROC associated with $\mathcal{L}_{\mathrm{u}}\{x\}$ is always a ***right-half plane***.
- For this reason, we often ***do not explicitly indicate the ROC*** when working with the unilateral Laplace transform.

- With the unilateral Laplace transform, the same inverse transform equation is used as in the bilateral case.
- The unilateral Laplace transform is ***only invertible for causal functions***. In particular, we have

$$\begin{aligned}
\mathcal{L}_{\mathsf{u}}^{-1}\{\mathcal{L}_{\mathsf{u}}\{x\}\}(t) &= \mathcal{L}_{\mathsf{u}}^{-1}\{\mathcal{L}\{xu\}\}(t) \\
&= \mathcal{L}^{-1}\{\mathcal{L}\{xu\}\}(t) \\
&= x(t)u(t) \\
&= \begin{cases} x(t) & t > 0 \\ 0 & t < 0. \end{cases}
\end{aligned}$$

- For a noncausal function x, we can only recover $x(t)$ for $t \geq 0$.

Unilateral Laplace Transform (Continued 2)

- Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties.
- Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways.

Properties of the Unilateral Laplace Transform

Property	Time Domain	Laplace Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$
Time/Laplace-Domain Scaling	$x(at)$, $a>0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1*x_2(t)$, x_1 and x_2 are causal	$X_1(s)X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$
Time-Domain Integration	$\int_{0^-}^{t}x(\tau)d\tau$	$\frac{1}{s}X(s)$

Property	
Initial Value Theorem	$x(0^+)=\lim_{s\to\infty}sX(s)$
Final Value Theorem	$\lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)$

Unilateral Laplace Transform Pairs

Pair	$x(t),\ t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos \omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin \omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at} \cos \omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

Solving Differential Equations Using the Unilateral Laplace Transform

- Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations.
- One common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions.

Part 8

Discrete-Time (DT) Signals and Systems

Section 8.1

Independent- and Dependent-Variable Transformations

Time Shifting (Translation)

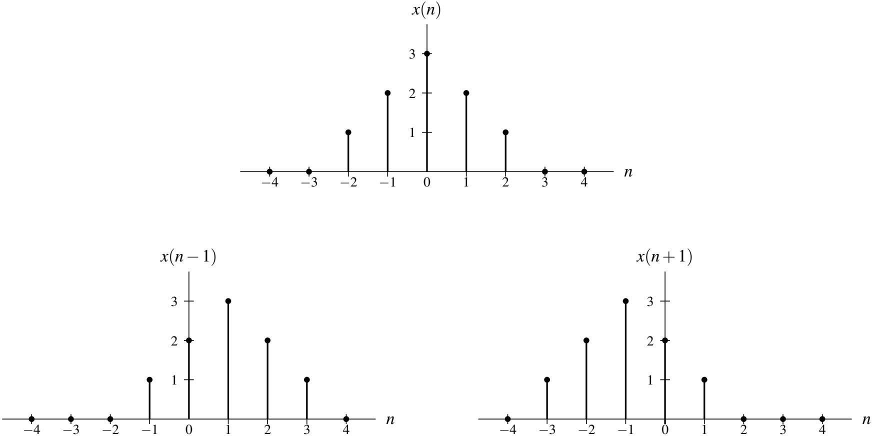
- **Time shifting** (also called **translation**) maps the input sequence x to the output sequence y as given by

$$y(n) = x(n-b),$$

where b is an integer.

- Such a transformation shifts the sequence (to the left or right) along the time axis.
- If $b > 0$, y is ***shifted to the right*** by $|b|$, relative to x (i.e., delayed in time).
- If $b < 0$, y is ***shifted to the left*** by $|b|$, relative to x (i.e., advanced in time).

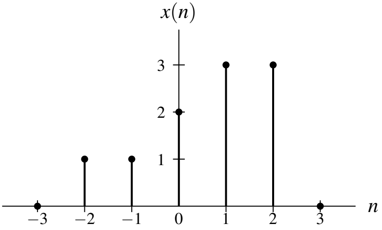
Time Shifting (Translation): Example

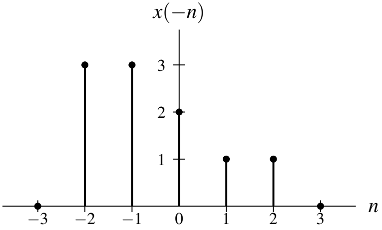


- **Time reversal** (also known as **reflection**) maps the input sequence x to the output sequence y as given by

$$y(n) = x(-n).$$

- Geometrically, the output sequence y is a reflection of the input sequence x about the (vertical) line $n = 0$.



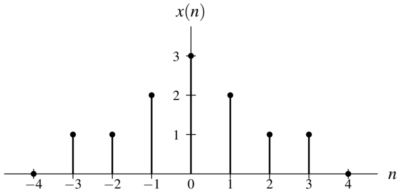


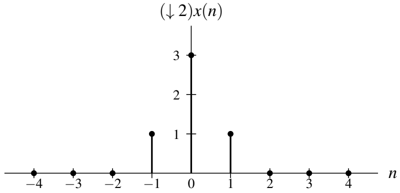
- **Downsampling** maps the input sequence x to the output sequence y as given by

$$y(n) = (\downarrow a)x(n) = x(an),$$

where a is a ***strictly positive*** integer.

- The output sequence y is produced from the input sequence x by keeping only every ath sample of x.



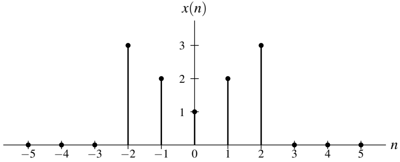


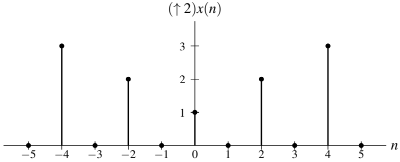
- **Upsampling** maps the input sequence x to the output sequence y as given by

$$y(n) = (\uparrow a)x(n) = \begin{cases} x(n/a) & n/a \text{ is an integer} \\ 0 & \text{otherwise,} \end{cases}$$

where a is a strictly positive integer.

- The output sequence y is produced from the input sequence x by inserting $a-1$ zeros between all of the samples of x.





Combined Independent-Variable Transformations

- Consider a transformation that maps the input sequence x to the output sequence y as given by

$$y(n) = x(an - b),$$

  where a and b are integers and $a \neq 0$.
- Such a transformation is a combination of time shifting, downsampling, and time reversal operations.
- Time reversal ***commutes*** with downsampling.
- Time shifting ***does not commute*** with time reversal or downsampling.
- The above transformation is equivalent to:
  1. first, time shifting x by b;
  2. then, downsampling the result by $|a|$ and, if $a < 0$, time reversing as well.
- If $\frac{b}{a}$ is an integer, the above transformation is also equivalent to:
  1. first, downsampling x by $|a|$ and, if $a < 0$, time reversing;
  2. then, time shifting the result by $\frac{b}{a}$.
- Note that the time shift is not by the same amount in both cases.

Section 8.2

Properties of Sequences

Symmetry and Addition/Multiplication

- Sums involving even and odd sequences have the following properties:
  - The sum of two even sequences is even.
  - The sum of two odd sequences is odd.
  - The sum of an even sequence and odd sequence is neither even nor odd, provided that neither of the sequences is identically zero.
- That is, the ***sum*** of sequences with the ***same type of symmetry*** also has the ***same type of symmetry***.
- Products involving even and odd sequences have the following properties:
  - The product of two even sequences is even.
  - The product of two odd sequences is even.
  - The product of an even sequence and an odd sequence is odd.
- That is, the ***product*** of sequences with the ***same type of symmetry*** is ***even***, while the ***product*** of sequences with ***opposite types of symmetry*** is ***odd***.

- Every sequence x has a ***unique*** representation of the form

$$x(n) = x_\mathsf{e}(n) + x_\mathsf{o}(n),$$

  where the sequences x_e and x_o are ***even*** and ***odd***, respectively.
- In particular, the sequences x_e and x_o are given by

$$x_\mathsf{e}(n) = \tfrac{1}{2}\left[x(n) + x(-n)\right] \quad \text{and} \quad x_\mathsf{o}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

- The sequences x_e and x_o are called the **even part** and **odd part** of x, respectively.
- For convenience, the even and odd parts of x are often denoted as $\mathrm{Even}\{x\}$ and $\mathrm{Odd}\{x\}$, respectively.

- The **least common multiple** of two (strictly positive) integers a and b, denoted $\operatorname{lcm}(a,b)$, is the smallest positive integer that is divisible by both a and b.
- The quantity $\operatorname{lcm}(a,b)$ can be easily determined from a prime factorization of the integers a and b by taking the product of the highest power for each prime factor appearing in these factorizations. Example:

$$\operatorname{lcm}(20,6) = \operatorname{lcm}(2^2 \cdot 5^1, 2^1 \cdot 3^1) = 2^2 \cdot 3^1 \cdot 5^1 = 60;$$
$$\operatorname{lcm}(54,24) = \operatorname{lcm}(2^1 \cdot 3^3, 2^3 \cdot 3^1) = 2^3 \cdot 3^3 = 216; \quad \text{and}$$
$$\operatorname{lcm}(24,90) = \operatorname{lcm}(2^3 \cdot 3^1, 2^1 \cdot 3^2 \cdot 5^1) = 2^3 \cdot 3^2 \cdot 5^1 = 360.$$

- **Sum of periodic sequences.** For any two periodic sequences x_1 and x_2 with fundamental periods N_1 and N_2, respectively, the sum $x_1 + x_2$ is ***periodic*** with period $\operatorname{lcm}(N_1, N_2)$.

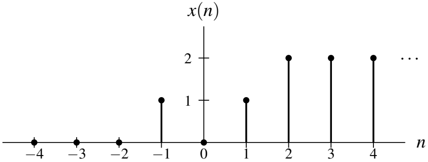
Right-Sided Sequences

- A sequence x is said to be **right sided** if, for some (finite) integer constant n_0, the following condition holds:

$$x(n) = 0 \quad \text{for all } n < n_0$$

(i.e., x is ***only potentially nonzero to the right of*** n_0).

- An example of a right-sided sequence is shown below.



- A sequence x is said to be **causal** if

$$x(n) = 0 \quad \text{for all } n < 0.$$

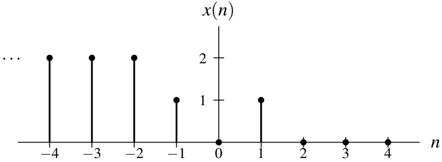
- A causal sequence is a ***special case*** of a right-sided sequence.
- A causal sequence is not to be confused with a causal system. In these two contexts, the word “causal” has very different meanings.

- A sequence x is said to be **left sided** if, for some (finite) integer constant n_0, the following condition holds:

$$x(n) = 0 \quad \text{for all } n > n_0$$

(i.e., x is ***only potentially nonzero to the left of*** n_0).

- An example of a left-sided sequence is shown below.



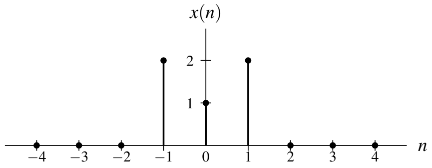
- A sequence x is said to be **anticausal** if

$$x(n) = 0 \quad \text{for all } n > 0.$$

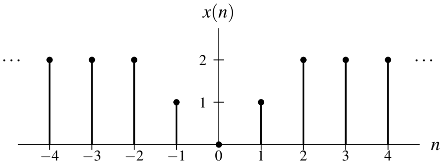
- An anticausal sequence is a ***special case*** of a left-sided sequence.
- An anticausal sequence is not to be confused with an anticausal system. In these two contexts, the word “anticausal” has very different meanings.

Finite-Duration and Two-Sided Sequences

- A sequence that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite-duration sequence is shown below.



- A sequence that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided sequence is shown below.



Bounded Sequences

- A sequence x is said to be **bounded** if there exists some (***finite***) positive real constant A such that

$$|x(n)| \leq A \quad \text{for all } n$$

  (i.e., $x(n)$ is ***finite*** for all n).
- Examples of bounded sequences include any constant sequence.
- Examples of unbounded sequences include any nonconstant polynomial sequence.

- The **energy** E contained in the sequence x is given by

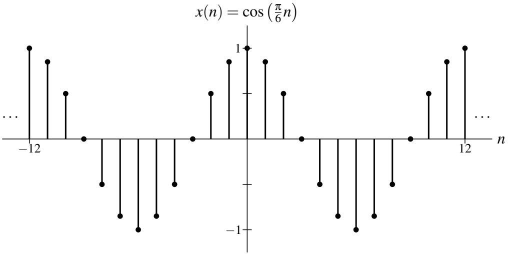
$$E = \sum_{k=-\infty}^{\infty} |x(k)|^2 .$$

- A signal with finite energy is said to be an **energy signal**.

Section 8.3

Elementary Sequences

- A **real sinusoidal sequence** is a sequence of the form
$$x(n) = A\cos(\Omega n + \theta),$$
where A, Ω, and θ are ***real*** constants.
- A real sinusoid is ***periodic*** if and only if $\frac{\Omega}{2\pi}$ is a ***rational number***, in which case the fundamental period is the ***smallest integer*** of the form $\frac{2\pi k}{|\Omega|}$ where k is a positive integer.
- For all integer k, $x_k(n) = A\cos([\Omega + 2\pi k]n + \theta)$ is the ***same*** sequence.
- An example of a periodic real sinusoid with fundamental period 12 is shown plotted below.



- A **complex exponential sequence** is a sequence of the form

$$x(n) = ca^n,$$

where c and a are ***complex*** constants.

- Such a sequence can also be equivalently expressed in the form

$$x(n) = ce^{bn},$$

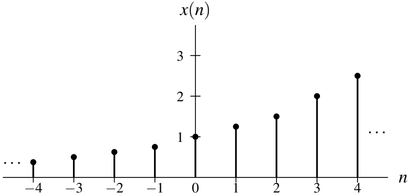
where b is a ***complex*** constant chosen as $b = \ln a$. (This this form is more similar to that presented for CT complex exponentials).

- A complex exponential can exhibit one of a number of ***distinct modes of behavior***, depending on the values of the parameters c and a.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

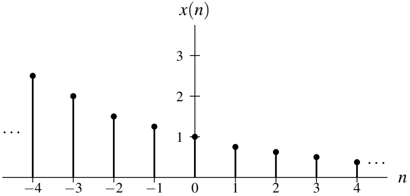
- A **real exponential sequence** is a special case of a complex exponential

$$x(n) = ca^n,$$

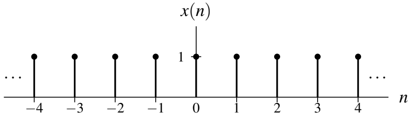
  where c and a are restricted to be ***real*** numbers.
- A real exponential can exhibit one of ***several distinct modes*** of behavior, depending on the magnitude and sign of a.
- If $|a| > 1$, the magnitude of $x(n)$ ***increases*** exponentially as n increases (i.e., a growing exponential).
- If $|a| < 1$, the magnitude of $x(n)$ ***decreases*** exponentially as n increases (i.e., a decaying exponential).
- If $|a| = 1$, the magnitude of $x(n)$ is a ***constant***, independent of n.
- If $a > 0$, $x(n)$ has the ***same sign*** for all n.
- If $a < 0$, $x(n)$ ***alternates in sign*** as n increases/decreases.



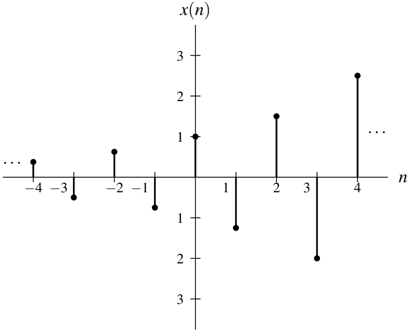
$|a| > 1,\ a > 0$ $[a = \frac{5}{4};\ c = 1]$



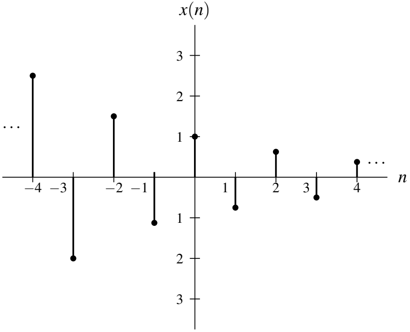
$|a| < 1,\ a > 0$ $[a = \frac{4}{5};\ c = 1]$



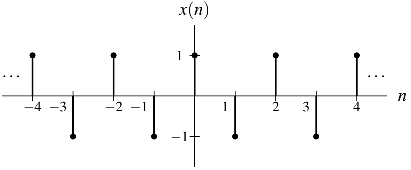
$|a| = 1,\ a > 0$ $[a = 1;\ c = 1]$



$|a| > 1, \ a < 0$ $[a = -\frac{5}{4}; c = 1]$



$|a| < 1, \ a < 0$ $[a = -\frac{4}{5}; c = 1]$



$|a| = 1, \ a < 0$ $[a = -1; c = 1]$

- A complex sinusoidal sequence is a special case of a complex exponential $x(n) = ca^n$, where c and a are ***complex*** and $|a| = 1$ (i.e., a is of the form $e^{j\Omega}$ where Ω is real).
- That is, a **complex sinusoidal sequence** is a sequence of the form

$$x(n) = ce^{j\Omega n},$$

where c is ***complex*** and Ω is ***real***.

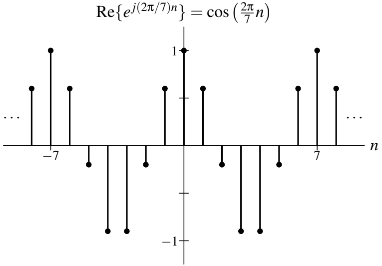
- Using Euler's relation, we can rewrite $x(n)$ as

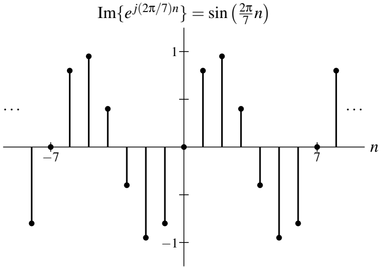
$$x(n) = \underbrace{|c|\cos(\Omega n + \arg c)}_{\mathrm{Re}\{x(n)\}} + j\underbrace{|c|\sin(\Omega n + \arg c)}_{\mathrm{Im}\{x(n)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are real sinusoids.
- A complex sinusoid is ***periodic*** if and only if $\frac{\Omega}{2\pi}$ is a ***rational number***, in which case the fundamental period is the ***smallest integer*** of the form $\frac{2\pi k}{|\Omega|}$ where k is a positive integer.

Complex Sinusoidal Sequences (Continued)

- For $x(n) = e^{j(2\pi/7)n}$, the graphs of $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are shown below.



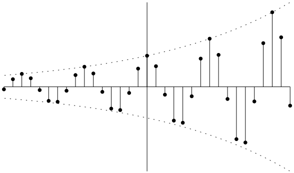


- In the most general case of a complex exponential sequence $x(n) = ca^n$, c and a are both ***complex***.
- Letting $c = |c| e^{j\theta}$ and $a = |a| e^{j\Omega}$ where θ and Ω are real, and using Euler's relation, we can rewrite $x(n)$ as

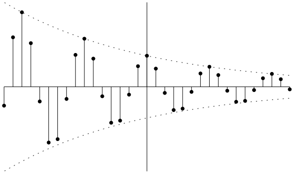
$$x(n) = \underbrace{|c| |a|^n \cos(\Omega n + \theta)}_{\mathrm{Re}\{x(n)\}} + j \underbrace{|c| |a|^n \sin(\Omega n + \theta)}_{\mathrm{Im}\{x(n)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real exponential and real sinusoid.
- One of ***several distinct modes*** of behavior is exhibited by x, depending on the value of a.
- If $|a| = 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are ***real sinusoids***.
- If $|a| > 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a growing real exponential***.
- If $|a| < 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a decaying real exponential***.

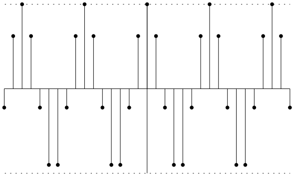
- The ***various modes of behavior*** for $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are illustrated below.



$|a| > 1$



$|a| < 1$



$|a| = 1$

Relationship Between Complex Exponentials and Real Sinusoids

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$ce^{j\Omega n} = c\cos\Omega n + jc\sin\Omega n.$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

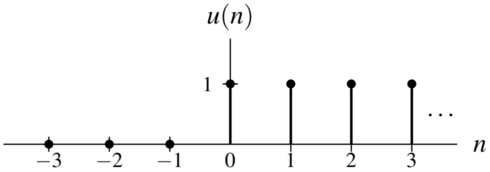
$$c\cos(\Omega n+\theta) = \frac{c}{2}\left[e^{j(\Omega n+\theta)} + e^{-j(\Omega n+\theta)}\right] \quad \text{and}$$
$$c\sin(\Omega n+\theta) = \frac{c}{2j}\left[e^{j(\Omega n+\theta)} - e^{-j(\Omega n+\theta)}\right].$$

- Note that, above, we are simply ***restating results*** from the (appendix) material on complex analysis.

- The **unit-step sequence**, denoted u, is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this sequence is shown below.



Unit Rectangular Pulses

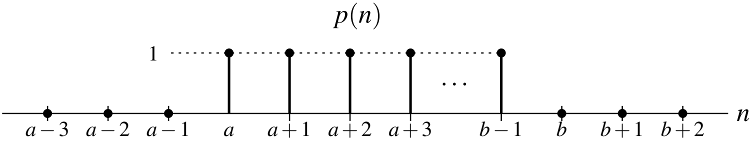
- A **unit rectangular pulse** is a sequence of the form

$$p(n) = \begin{cases} 1 & a \leq n < b \\ 0 & \text{otherwise} \end{cases}$$

  where a and b are integer constants satisfying $a < b$.
- Such a sequence can be expressed in terms of the unit-step sequence as

$$p(n) = u(n-a) - u(n-b).$$

- The graph of a unit rectangular pulse has the general form shown below.



- The **unit-impulse sequence** (also known as the **delta sequence**), denoted δ, is defined as

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

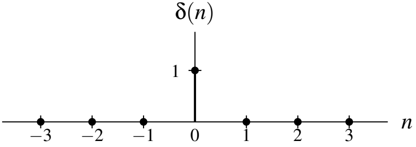
- The first-order difference of u is δ. That is,

$$\delta(n) = u(n) - u(n-1).$$

- The running sum of δ is u. That is,

$$u(n) = \sum_{k=-\infty}^{n} \delta(k).$$

- A plot of δ is shown below.



Properties of the Unit-Impulse Sequence

- For any sequence x and any integer constant n_0, the following identity holds:

$$x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0).$$

- For any sequence x and any integer constant n_0, the following identity holds:

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0).$$

- Trivially, the sequence δ is also even.

Section 8.4

Discrete-Time (DT) Systems

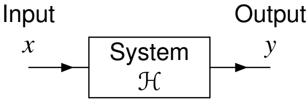
DT Systems

- A system with input x and output y can be described by the equation
$$y = \mathcal{H}x,$$
where $\mathcal{H}$ denotes an operator (i.e., transformation).
- Note that the operator $\mathcal{H}$ ***maps a sequence to a sequence*** (not a number to a number).
- Alternatively, we can express the above relationship using the notation
$$x \xrightarrow{\mathcal{H}} y.$$
- If clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation
$$x \to y.$$
- Note that the symbols "$\to$" and "=" have ***very different*** meanings.
- The symbol "$\to$" should be read as "***produces***" (not as "equals").

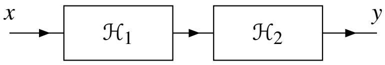
Remarks on Operator Notation for Systems

- For a system operator $\mathcal{H}$ and a sequence x, $\mathcal{H}x$ is the sequence produced as the output of the system $\mathcal{H}$ when the input is the sequence x.
- Brackets around the operand of an operator are ***usually omitted when not required*** for grouping.
- For example, for an operator $\mathcal{H}$, a sequence x, and an integer n, we would normally prefer to write:
  1. $\mathcal{H}x$ instead of the equivalent expression $\mathcal{H}(x)$; and
  2. $\mathcal{H}x(n)$ instead of the equivalent expression $\mathcal{H}(x)(n)$.
- Also, note that $\mathcal{H}x$ is a ***sequence*** and $\mathcal{H}x(n)$ is a ***number*** (namely, the value of the sequence $\mathcal{H}x$ evaluated at the index n).
- In the expression $\mathcal{H}(x_1 + x_2)$, the brackets are needed for grouping, since $\mathcal{H}(x_1 + x_2) \not\equiv \mathcal{H}x_1 + x_2$ (where "$\not\equiv$" means "not equivalent").
- When multiple operators are applied, they group from ***right to left***.
- For example, for the operators $\mathcal{H}_1$ and $\mathcal{H}_2$, and the sequence x, the expression $\mathcal{H}_2\mathcal{H}_1x$ means $\mathcal{H}_2[\mathcal{H}_1(x)]$.

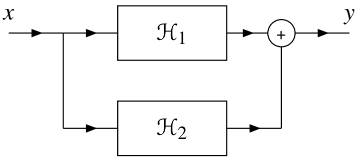
- Often, a system defined by the operator $\mathcal{H}$ and having the input x and output y is represented in the form of a ***block diagram*** as shown below.



- ***Two basic ways*** in which systems can be interconnected are shown below.



Series



Parallel

- A **series** (or **cascade**) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = \mathcal{H}_2\mathcal{H}_1 x.$$

- A **parallel** connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x.$$

Section 8.5

Properties of (DT) Systems

- A system $\mathcal{H}$ is said to be **memoryless** if, for every integer constant n_0, $\mathcal{H}x(n_0)$ does not depend on $x(n)$ for some $n \neq n_0$.
- In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the ***same*** point in time.
- A system that is not memoryless is said to have **memory**.
- Although simple, a memoryless system is ***not very flexible***, since its current output value cannot rely on past or future values of the input.

Causality

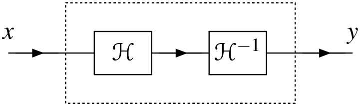
- A system $\mathcal{H}$ is said to be **causal** if, for every integer constant n_0, $\mathcal{H}x(n_0)$ does not depend on $x(n)$ for some $n > n_0$.
- In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the ***same or earlier points*** in time (i.e., ***not later points in time***).
- If the independent variable n represents time, a system must be causal in order to be ***physically realizable***.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable ***need not always represent time*** (e.g., the independent variable might represent position).
- A memoryless system is always causal, although the converse is not necessarily true.

Invertibility

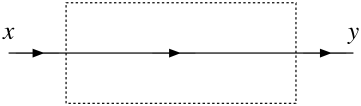
- The **inverse** of a system $\mathcal{H}$ is another system $\mathcal{H}^{-1}$ such that, for every sequence x,
$$\mathcal{H}^{-1}\mathcal{H}x = x$$
(i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{H}^{-1}$ is a system whose input and output are equal).
- A system is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input x can always be ***uniquely*** determined from its output y.
- An invertible system will always produce ***distinct outputs*** from any two ***distinct inputs***.
- To show that a system is ***invertible***, we simply find the ***inverse system***.
- To show that a system is ***not invertible***, we find ***two distinct inputs*** that result in ***identical outputs***.
- In practical terms, invertible systems are "nice" in the sense that their ***effects can be undone***.

Invertibility (Continued)

- A system $\mathcal{H}^{-1}$ being the inverse of $\mathcal{H}$ means that the following two systems are equivalent (i.e., $\mathcal{H}^{-1}\mathcal{H}$ ***is an identity***):



System 1: $y = \mathcal{H}^{-1}\mathcal{H}x$



System 2: $y = x$

Bounded-Input Bounded-Output (BIBO) Stability

- A system $\mathcal{H}$ is **BIBO stable** if, for every bounded sequence x, $\mathcal{H}x$ is bounded (i.e., $|x(n)| < \infty$ for all n implies that $|\mathcal{H}x(n)| < \infty$ for all n).
- In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.
- To show that a system is ***BIBO stable***, we must show that ***every bounded input*** leads to a ***bounded output***.
- To show that a system is ***not BIBO stable***, we need only find a single ***bounded input*** that leads to an ***unbounded output***.
- In practical terms, a BIBO stable system is ***well behaved*** in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time.
- Usually, a system that is not BIBO stable will have ***serious safety issues***.
- For example, a portable music player with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized human (and likely a big lawsuit as well).

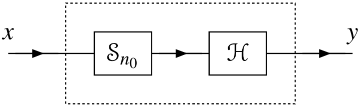
- A system $\mathcal{H}$ is said to be **time invariant (TI)** if, for every sequence x and every integer n_0, the following condition holds:

$$\mathcal{H}x(n-n_0) = \mathcal{H}x'(n) \text{ for all } n, \quad \text{where} \quad x'(n) = x(n-n_0)$$

(i.e., $\mathcal{H}$ ***commutes with time shifts***).

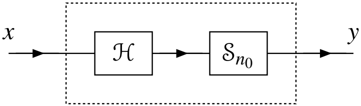
- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an ***identical time shift*** in the output.
- A system that is not time invariant is said to be **time varying**.
- In simple terms, a time invariant system is a system whose behavior ***does not change*** with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much ***easier to design and analyze***, since their behavior does not change with respect to time.

- Let $\mathcal{S}_{n_0}$ denote an operator that applies a ***time shift of*** n_0 to a sequence (i.e., $\mathcal{S}_{n_0}x(n) = x(n-n_0)$).
- A system $\mathcal{H}$ is ***time invariant*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with*** $\mathcal{S}_{n_0}$):



System 1: $y = \mathcal{H}\mathcal{S}_{n_0}x$

$$\begin{bmatrix} y(n) = \mathcal{H}x'(n) \\ x'(n) = \mathcal{S}_{n_0}x(n) = x(n-n_0) \end{bmatrix}$$



System 2: $y = \mathcal{S}_{n_0}\mathcal{H}x$

$$\left[y(n) = \mathcal{H}x(n-n_0)\right]$$

- A system $\mathcal{H}$ is said to be **additive** if, for all sequences x_1 and x_2, the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

  (i.e., $\mathcal{H}$ ***commutes with sums***).
- A system $\mathcal{H}$ is said to be **homogeneous** if, for every sequence x and every complex constant a, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

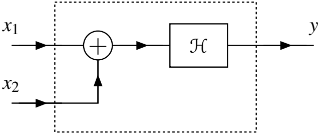
  (i.e., $\mathcal{H}$ ***commutes with multiplication by a constant***).
- A system that is both additive and homogeneous is said to be **linear**.
- In other words, a system $\mathcal{H}$ is ***linear***, if for all sequences x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

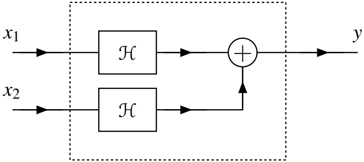
  (i.e., $\mathcal{H}$ ***commutes with linear combinations***).
- The linearity property is also referred to as the **superposition** property.
- Practically speaking, linear systems are much ***easier to design and analyze*** than nonlinear systems.

Additivity, Homogeneity, and Linearity (Continued 1)

- The system $\mathcal{H}$ is ***additive*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with addition***):

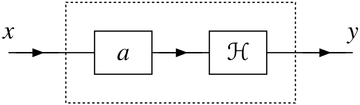


System 1: $y = \mathcal{H}(x_1 + x_2)$

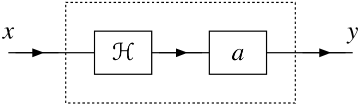


System 2: $y = \mathcal{H}x_1 + \mathcal{H}x_2$

- The system $\mathcal{H}$ is ***homogeneous*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with scalar multiplication***):

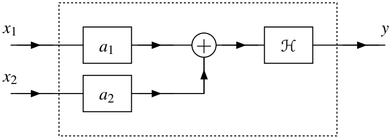


System 1: $y = \mathcal{H}(ax)$

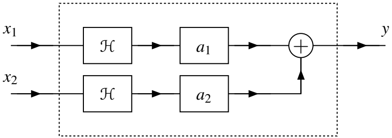


System 2: $y = a\mathcal{H}x$

- The system $\mathcal{H}$ is ***linear*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with linear combinations***):



System 1: $y = \mathcal{H}(a_1x_1 + a_2x_2)$



System 2: $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$

- A sequence x is said to be an **eigensequence** of the system $\mathcal{H}$ with the **eigenvalue** λ if

$$\mathcal{H}x = \lambda x,$$

  where λ is a complex constant.
- In other words, the system $\mathcal{H}$ acts as an ideal amplifier for each of its eigensequences x, where the amplifier gain is given by the corresponding eigenvalue λ.
- Different systems have different eigensequences.
- Many of the mathematical tools developed for the study of DT systems have eigensequences as their basis.

Part 9

Discrete-Time Linear Time-Invariant (LTI) Systems

Why Linear Time-Invariant (LTI) Systems?

- In engineering, linear time-invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Section 9.1

Convolution

DT Convolution

- The (DT) **convolution** of the sequences x and h, denoted $x * h$, is defined as the sequence

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

- The convolution $x * h$ evaluated at the point n is simply a weighted sum of elements of x, where the weighting is given by h time reversed and shifted by n.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in the theory of (DT) systems.
- In particular, convolution has a special significance in the context of (DT) LTI systems.

- To compute the convolution

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k),$$

we proceed as follows:

1. Plot $x(k)$ and $h(n-k)$ as a function of k.
2. Initially, consider an arbitrarily large negative value for n. This will result in $h(n-k)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $x * h(n)$.
4. Increase n gradually until the expression for $x * h(n)$ changes form. Record the interval over which the expression for $x * h(n)$ was valid.
5. Repeat steps 3 and 4 until n is an arbitrarily large positive value. This corresponds to $h(n-k)$ being shifted very far to the right on the time axis.
6. The results for the various intervals can be combined in order to obtain an expression for $x * h(n)$ for all n.

- The convolution operation is ***commutative***. That is, for any two sequences x and h,

$$x * h = h * x.$$

- The convolution operation is ***associative***. That is, for any sequences x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is ***distributive*** with respect to addition. That is, for any sequences x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

- For any sequence x,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) = x * \delta(n).$$

- Thus, any sequence x can be written in terms of an expression involving δ.
- Moreover, δ is the ***convolutional identity***. That is, for any sequence x,

$$x * \delta = x.$$

Circular Convolution

- The convolution of two periodic sequences is usually not well defined.
- This motivates an alternative notion of convolution for periodic sequences known as circular convolution.
- The **circular convolution** (also known as the DT periodic convolution) of the N-periodic sequences x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(n) = \sum_{k=\langle N \rangle} x(k)h(n-k) = \sum_{k=0}^{N-1} x(k)h(\operatorname{mod}(n-k,N)),$$

  where $\operatorname{mod}(a,b)$ is the remainder after division when a is divided by b.
- The circular convolution and (linear) convolution of the N-periodic sequences x and h are related as follows:

$$x \circledast h(n) = x_0 * h(n) \quad \text{where} \quad x(n) = \sum_{k=-\infty}^{\infty} x_0(n-kN)$$

  (i.e., $x_0(n)$ equals $x(n)$ over a single period of x and is zero elsewhere).

Section 9.2

Convolution and LTI Systems

Impulse Response

- The response h of a system $\mathcal{H}$ to the input δ is called the **impulse response** of the system (i.e., $h = \mathcal{H}\delta$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

- In other words, a LTI system simply ***computes a convolution***.
- Furthermore, a LTI system is ***completely characterized*** by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.

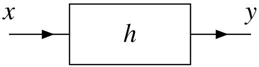
Step Response

- The response s of a system $\mathcal{H}$ to the input u is called the **step response** of the system (i.e., $s = \mathcal{H}u$).
- The impulse response h and step response s of a system are related as

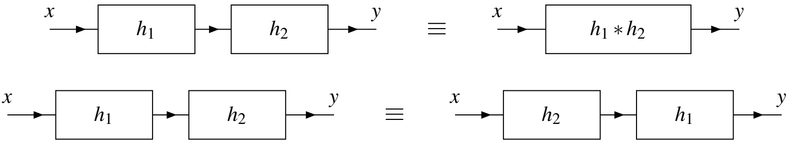
$$h(n) = s(n) - s(n-1).$$

- Therefore, the impulse response of a system can be determined from its step response by (first-order) differencing.

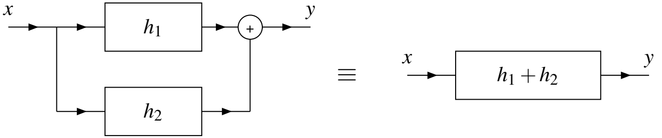
- Often, it is convenient to represent a (DT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.



- The ***series*** interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h = h_1 * h_2$. That is, we have the equivalences shown below.



- The ***parallel*** interconnection of the LTI systems with impulse responses h_1 and h_2 is a LTI system with the impulse response $h = h_1 + h_2$. That is, we have the equivalence shown below.



Section 9.3

Properties of LTI Systems

Memory

- A LTI system with impulse response h is memoryless if and only if

$$h(n) = 0 \quad \text{for all } n \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(n) = K\delta(n),$$

where K is a complex constant.

- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\delta) = Kx$$

(i.e., the system is an ideal amplifier).

- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

Causality

- A LTI system with impulse response h is causal if and only if

$$h(n) = 0 \quad \text{for all } n < 0$$

  (i.e., h is a causal sequence).

- It is due to the above relationship that we call a sequence x, satisfying

$$x(n) = 0 \quad \text{for all } n < 0,$$

  a causal sequence.

Invertibility

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{\text{inv}} = \delta.$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a sequence h_{inv} such that

$$h * h_{\text{inv}} = \delta.$$

- Except in simple cases, the above condition is often quite difficult to test.

- A LTI system with impulse response h is BIBO stable if and only if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

(i.e., h is ***absolutely summable***).

Eigensequences of LTI Systems

- As it turns out, every complex exponential is an eigensequence of all LTI systems.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{z^n\}(n) = H(z)z^n,$$

  where z is a complex constant and

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}.$$

- That is, z^n is an eigensequence of a LTI system and $H(z)$ is the corresponding eigenvalue.
- We refer to H as the **system function** (or **transfer function**) of the system $\mathcal{H}$.
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(z)$.

Representation of Sequences Using Eigensequences

- Consider a LTI system with input x, output y, and system function H.
- Suppose that the input x can be expressed as the linear combination of complex exponentials
$$x(n) = \sum_k a_k z_k^n,$$
where the a_k and z_k are complex constants.
- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude
$$y(n) = \sum_k a_k H(z_k) z_k^n.$$
- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as linear combination of the ***same*** complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Part 10

Discrete-Time Fourier Series (DTFS)

- The Fourier series is a representation for ***periodic*** sequences.
- With a Fourier series, a sequence is represented as a ***linear combination of complex sinusoids***.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- Perhaps, most importantly, complex sinusoids are ***eigensequences*** of (DT) LTI systems.

Section 10.1

Fourier Series

- A set of periodic complex sinusoids is said to be **harmonically related** if there exists some constant $2\pi/N$ such that the fundamental frequency of each complex sinusoid is an integer multiple of $2\pi/N$.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(n) = e^{j(2\pi/N)kn} \quad \text{for all integer } k.$$

- In the above set $\{\phi_k\}$, only N elements are distinct, since

$$\phi_k = \phi_{k+N} \quad \text{for all integer } k.$$

- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of $\frac{2\pi}{N}$, a linear combination of these complex sinusoids must be N-periodic.

- A periodic complex-valued sequence x with fundamental period N can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn},$$

where $\sum_{k=\langle N \rangle}$ denotes summation over any N consecutive integers (e.g., $0, 1, \ldots, N-1$). (The summation can be taken over any N consecutive integers, due to the N-periodic nature of x and $e^{j(2\pi/N)kn}$.)
- The above representation of x is known as the (DT) **Fourier series** and the a_k are called **Fourier series coefficients**.
- The above formula for x is often called the **Fourier series synthesis equation**.
- The terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K(2\pi/N)$.
- To denote that the sequence x has the Fourier series coefficient sequence a, we write

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k.$$

DT Fourier Series (DTFS) (Continued)

- A periodic sequence x with fundamental period N has the Fourier series coefficient sequence a given by

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

(The summation can be taken over any N consecutive integers due to the N-periodic nature of x and $e^{-j(2\pi/N)kn}$.)

- The above equation for a_k is often referred to as the **Fourier series analysis equation**.
- Due to the N-periodic nature of x and $e^{-j(2\pi/N)kn}$, the sequence a is also N-periodic.

- Consider the N-periodic sequence x with Fourier series coefficient sequence a.
- If x is real, then its Fourier series can be rewritten in trigonometric form as shown below.
- The **trigonometric form** of a Fourier series has the appearance

$$x(n) = \begin{cases} \alpha_0 + \displaystyle\sum_{k=1}^{N/2-1} \left[\alpha_k \cos\left(\tfrac{2\pi kn}{N}\right) + \beta_k \sin\left(\tfrac{2\pi kn}{N}\right)\right] + & \\ \quad \alpha_{N/2} \cos \pi n & N \text{ even} \\ \alpha_0 + \displaystyle\sum_{k=1}^{(N-1)/2} \left[\alpha_k \cos\left(\tfrac{2\pi kn}{N}\right) + \beta_k \sin\left(\tfrac{2\pi kn}{N}\right)\right] & N \text{ odd,} \end{cases}$$

where $\alpha_0 = a_0$, $\alpha_{N/2} = a_{N/2}$, $\alpha_k = 2\operatorname{Re} a_k$, and $\beta_k = -2\operatorname{Im} a_k$.
- Note that the above trigonometric form contains only ***real*** quantities.

Prelude to the Discrete Fourier Transform (DFT)

- Letting $a'_k = Na_k$, we can rewrite the Fourier series synthesis and analysis equations, respectively, as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} a'_k e^{j(2\pi/N)kn} \quad \text{and} \quad a'_k = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}.$$

- Since x and a' are both N-periodic, each of these sequences is completely characterized by its N samples over a single period.
- If we only consider the behavior of x and a' over a single period, this leads to the equations

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} a'_k e^{j(2\pi/N)kn} \quad \text{for } n = 0,1,\ldots,N-1 \quad \text{and}$$

$$a'_k = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} \quad \text{for } k = 0,1,\ldots,N-1.$$

- As it turns out, the above two equations define what is known as the discrete Fourier transform (DFT).

Discrete Fourier Transform (DFT)

- The **discrete Fourier transform (DFT)** X of the sequence x is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} \quad \text{for } k = 0, 1, \ldots N-1.$$

- The preceding equation is known as the **DFT analysis equation**.
- The **inverse DFT** x of the sequence X is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)kn} \quad \text{for } n = 0, 1, \ldots N-1.$$

- The preceding equation is known as the **DFT synthesis equation**.
- The DFT maps a finite-length sequence of N samples to another finite-length sequence of N samples.
- The DFT will be considered in more detail later.

- Since the analysis and synthesis equations for (DT) Fourier series involve only ***finite*** sums (as opposed to infinite series), convergence is not a significant issue of concern.
- If an N-periodic sequence is bounded (i.e., is finite in value), its Fourier series coefficient sequence will exist and be bounded and the Fourier series analysis and synthesis equations must converge.

Section 10.2

Properties of Fourier Series

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(n) + \beta y(n)$	$\alpha a_k + \beta b_k$
Translation	$x(n - n_0)$	$e^{-jk(2\pi/N)n_0} a_k$
Modulation	$e^{j(2\pi/N)k_0 n} x(n)$	a_{k-k_0}
Reflection	$x(-n)$	a_{-k}
Conjugation	$x^*(n)$	a^*_{-k}
Duality	a_n	$\frac{1}{N} x(-k)$
Periodic Convolution	$x \circledast y(n)$	$N a_k b_k$
Multiplication	$x(n)y(n)$	$a \circledast b_k$

Property	
Parseval's Relation	$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x(n) \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

Linearity

- Let x and y be N-periodic sequences. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k$ and $y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k$, then

$$\alpha x(n) + \beta y(n) \overset{\text{DTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$

  where α and β are complex constants.
- That is, a linear combination of sequences produces the same linear combination of their Fourier series coefficients.

Translation (Time Shifting)

- Let x denote a periodic sequence with period N. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k$, then

$$x(n-n_0) \overset{\text{DTFS}}{\longleftrightarrow} e^{-jk(2\pi/N)n_0} c_k,$$

  where n_0 is an integer constant.

- In other words, time shifting a periodic sequence changes the argument (but not magnitude) of its Fourier series coefficients.

Modulation (Frequency Shifting)

- Let x denote a periodic sequence with period N. If $x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} c_k$, then

$$e^{j(2\pi/N)k_0 n} x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} c_{k-k_0},$$

  where k_0 is an integer constant.

- That is, multiplying a sequence by a complex sinusoid whose frequency is an integer multiple of $2\pi/N$ results in a translation of the corresponding Fourier series coefficient sequence.

- Let x denote a periodic sequence with period N. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k$, then

$$x(-n) \overset{\text{DTFS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversing a sequence results in a time reversal of the corresponding Fourier series coefficient sequence.

Conjugation

- Let x denote a periodic sequence with period N. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k$, then
$$x^*(n) \overset{\text{DTFS}}{\longleftrightarrow} c^*_{-k}.$$
- In other words, conjugating a sequence has the effect of time reversing and conjugating the corresponding Fourier series coefficient sequence.

Duality

- Let x denote a periodic sequence with period N. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} a(k)$, then

$$a(n) \overset{\text{DTFS}}{\longleftrightarrow} \tfrac{1}{N}x(-k).$$

- This is known as the **duality property** of Fourier series.
- This property follows from the high degree of symmetry in the analysis and synthesis Fourier-series equations, which are respectively given by

$$x(m) = \sum_{\ell=\langle N\rangle} a(\ell)e^{j(2\pi/N)\ell m} \quad \text{and} \quad a(m) = \frac{1}{N}\sum_{\ell=\langle N\rangle} x(\ell)e^{-j(2\pi/N)m\ell}.$$

- That is, the analysis and synthesis equations are identical except for a ***factor of N*** and ***different sign*** in the parameter for the exponential function.

Periodic Convolution

- Let x and y be N-periodic sequences. If $x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k$ and $y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k$, then

$$x \circledast y(n) \overset{\text{DTFS}}{\longleftrightarrow} N a_k b_k.$$

- That is, periodic convolution of two sequences multiplies their corresponding Fourier series coefficient sequences (up to a scale factor).

- Let x and y be N-periodic sequences. If $x(n) \xleftrightarrow{\text{DTFS}} a_k$ and $y(n) \xleftrightarrow{\text{DTFS}} b_k$, then

$$x(n)y(n) \xleftrightarrow{\text{DTFS}} a \circledast b(k).$$

- That is, multiplying two sequences results in a circular convolution of their corresponding Fourier series coefficient sequences.

Parseval's Relation

- A sequence x and its Fourier series coefficient sequence a satisfy the following relationship:

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2=\sum_{k=\langle N\rangle}|a_k|^2.$$

- The above relationship is simply stating that the amount of energy in a single period of x and the amount of energy in a single period of a are equal up to a scale factor.
- In other words, the transformation between a sequence and its Fourier series coefficient sequence preserves energy (up to a scale factor).

Even/Odd Symmetry

- For an N-periodic sequence x with Fourier-series coefficient sequence a, the following properties hold:

$$x \text{ is even} \Leftrightarrow a \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow a \text{ is odd}.$$

- In other words, the even/odd symmetry properties of x and a always match.

Real Sequences

- A sequence x is ***real*** if and only if its Fourier series coefficient sequence a satisfies

$$a_k = a^*_{-k} \text{ for all } k$$

(i.e., a is ***conjugate symmetric***).

- From properties of complex numbers, one can show that $a_k = a^*_{-k}$ is equivalent to

$$|a_k| = |a_{-k}| \quad \text{and} \quad \arg a_k = -\arg a_{-k}$$

(i.e., $|a_k|$ is ***even*** and $\arg a_k$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that a is real.

- For an N-periodic sequence x with Fourier-series coefficient sequence a, the following properties hold:
  1. a_0 is the average value of x over a single period;
  2. x is real and even $\Leftrightarrow$ a is real and even; and
  3. x is real and odd $\Leftrightarrow$ a is purely imaginary and odd.

Section 10.3

Fourier Series and Frequency Spectra

- The Fourier series provides us with an entirely new way to view sequences.
- Instead of viewing a sequence as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a sequence as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- This so called frequency-domain perspective is of fundamental importance in engineering.
- Many engineering problems can be solved ***much more easily*** using the frequency domain than the time domain.
- The Fourier series coefficients of a sequence x provide a means to ***quantify*** how much information x has at different frequencies.
- The distribution of information in a sequence over different frequencies is referred to as the ***frequency spectrum*** of the sequence.

- To gain further insight into the role played by the Fourier series coefficients a_k in the context of the frequency spectrum of the N-periodic sequence x, it is helpful to write the Fourier series with the a_k expressed in ***polar form*** as

$$x(n) = \sum_{k=0}^{N-1} a_k e^{j\Omega_0 kn} = \sum_{k=0}^{N-1} |a_k| e^{j(\Omega_0 kn + \arg a_k)},$$

  where $\Omega_0 = \frac{2\pi}{N}$.
- Clearly, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\Omega_0$ that has been ***amplitude scaled*** by a factor of $|a_k|$ and ***time-shifted*** by an amount that depends on $\arg a_k$.
- For a given k, the ***larger*** $|a_k|$ is, the larger is the amplitude of its corresponding complex sinusoid $e^{jk\Omega_0 n}$, and therefore the ***larger the contribution*** the kth term (which is associated with frequency $k\Omega_0$) will make to the overall summation.
- In this way, we can use $|a_k|$ as a ***measure*** of how much information a sequence x has at the frequency $k\Omega_0$.

- The Fourier series coefficients a_k of the sequence x are referred to as the **frequency spectrum** of x.
- The magnitudes $|a_k|$ of the Fourier series coefficients a_k are referred to as the **magnitude spectrum** of x.
- The arguments $\arg a_k$ of the Fourier series coefficients a_k are referred to as the **phase spectrum** of x.
- The frequency spectrum a_k of an N-periodic sequence is N-periodic in the coefficient index k and 2π-periodic in the frequency $\Omega = k\Omega_0$.
- The range of frequencies between $-\pi$ and π are referred to as the **baseband**.
- Often, the spectrum of a sequence is plotted against frequency $\Omega = k\Omega_0$ (over the single 2π period of the baseband) instead of the Fourier series coefficient index k.

- Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, the frequency spectrum is ***discrete*** in the independent variable (i.e., frequency).
- Due to the general appearance of frequency-spectrum plot (i.e., a number of vertical lines at various frequencies), we refer to such spectra as **line spectra**.

Section 10.4

Fourier Series and LTI Systems

Frequency Response

- Recall that a LTI system $\mathcal{H}$ with impulse response h is such that $\mathcal{H}\{z^n\}(n) = H(z)z^n$, where $H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$. (That is, complex exponentials are ***eigensequences*** of LTI systems.)
- Since a complex sinusoid is a ***special case*** of a complex exponential, we can reuse the above result for the special case of complex sinusoids.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\left\{e^{j\Omega n}\right\}(n) = H(e^{j\Omega})e^{j\Omega n},$$

  where Ω is real and

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}.$$

- That is, $e^{j\Omega n}$ is an ***eigensequence*** of a LTI system and $H(e^{j\Omega})$ is the corresponding ***eigenvalue***.
- The function $H(e^{j\Omega})$ is 2π-***periodic***, since $e^{j\Omega}$ is 2π-periodic.
- We refer to $H(e^{j\Omega})$ as the **frequency response** of the system $\mathcal{H}$.

- Consider a LTI system with input x, output y, and frequency response $H(e^{j\Omega})$.
- Suppose that the N-periodic input x is expressed as the Fourier series

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n}, \quad \text{where } \Omega_0 = \tfrac{2\pi}{N}.$$

- Using our knowledge about the ***eigensequences*** of LTI systems, we can conclude

$$y(n) = \sum_{k=0}^{N-1} a_k H(e^{jk\Omega_0}) e^{jk\Omega_0 n}.$$

- Thus, if the input x to a LTI system is a Fourier series, the output y is also a Fourier series. More specifically, if $x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k$ then $y(n) \overset{\text{DTFS}}{\longleftrightarrow} H(e^{jk\Omega_0})a_k$.
- The above formula can be used to determine the output of a LTI system from its input in a way that ***does not require convolution***.

Filtering

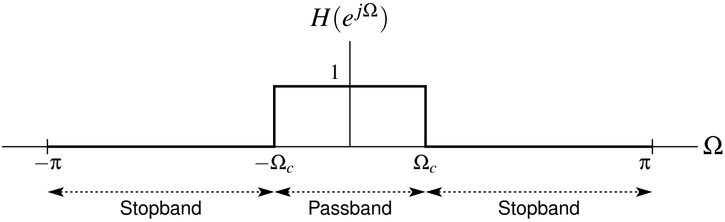
- In many applications, we want to ***modify the spectrum*** of a sequence by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a sequence is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

Ideal Lowpass Filter

- An **ideal lowpass filter** eliminates all baseband frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining baseband frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \le \Omega_c \\ 0 & \Omega_c < |\Omega| \le \pi, \end{cases}$$

  where Ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

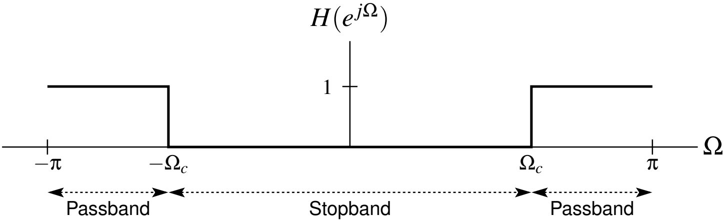


Ideal Highpass Filter

- An **ideal highpass filter** eliminates all baseband frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining baseband frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(e^{j\Omega}) = \begin{cases} 1 & \Omega_c < |\Omega| \leq \pi \\ 0 & |\Omega| \leq \Omega_c, \end{cases}$$

where Ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.



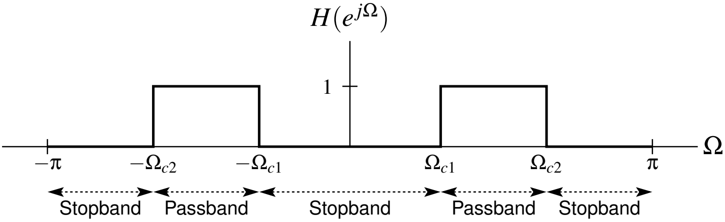
Ideal Bandpass Filter

- An **ideal bandpass filter** eliminates all baseband frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining baseband frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(e^{j\Omega}) = \begin{cases} 1 & \Omega_{c1} \leq |\Omega| \leq \Omega_{c2} \\ 0 & |\Omega| < \Omega_{c1} \text{ or } \Omega_{c2} < |\Omega| < \pi, \end{cases}$$

where the limits of the passband are Ω_{c1} and Ω_{c2}.

- A plot of this frequency response is given below.



Part 11

Discrete-Time Fourier Transform (DTFT)

Motivation for the Fourier Transform

- The (DT) Fourier series provide an extremely useful representation for periodic sequences.
- Often, however, we need to deal with sequences that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The Fourier transform can be used to represent both periodic and aperiodic sequences.
- Since the (DT) Fourier transform is essentially derived from (DT) Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Section 11.1

Fourier Transform

- The (DT) Fourier series is an extremely useful signal representation.
- Unfortunately, this signal representation can only be used for periodic sequences, since a Fourier series is inherently periodic.
- Many sequences are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can also be applied to aperiodic sequences.
- By viewing an aperiodic sequence as the limiting case of an N-periodic sequence where $N \to \infty$, we can use the Fourier series to develop a more general signal representation that can be used for both aperiodic and periodic sequences.
- This more general signal representation is called the (DT) Fourier transform.

- Recall that the Fourier series representation of a N-periodic sequence x is given by

$$x(n) = \sum_{k=\langle N \rangle} \underbrace{\left(\frac{1}{N} \sum_{\ell=\langle N \rangle} x(\ell) e^{-j(2\pi/N)k\ell} \right)}_{c_k} e^{j(2\pi/N)kn}.$$

- In the above representation, if we take the limit as $N \to \infty$, we obtain

$$x(n) = \frac{1}{2\pi} \int_{2\pi} \underbrace{\left(\sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \right)}_{X(\Omega)} e^{j\Omega n} d\Omega$$

(i.e., as $N \to \infty$, the two finite summations become an integral and infinite summation).

- This more general function representation is known as the Fourier transform representation.

- The **Fourier transform** of the sequence x, denoted $\mathcal{F}x$ or X, is given by

$$\mathcal{F}x(\Omega) = X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

- The preceding equation is sometimes referred to as **Fourier transform analysis equation** (or **forward Fourier transform equation**).
- The **inverse Fourier transform** of X, denoted $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(n) = x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega.$$

- The preceding equation is sometimes referred to as the **Fourier transform synthesis equation** (or **inverse Fourier transform equation**).
- As a matter of notation, to denote that a sequence x has the Fourier transform X, we write $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$.
- A sequence x and its Fourier transform X constitute what is called a **Fourier transform pair**.

Section 11.2

Convergence Properties of the Fourier Transform

- For a sequence x, the Fourier transform analysis equation (i.e., $X(\Omega) = \sum_{-\infty}^{\infty} x(n)e^{-j\Omega n}$) converges ***uniformly*** if

$$\sum_{k=-\infty}^{\infty} |x(k)| < \infty$$

(i.e., x is ***absolutely summable***).

- For a sequence x, the Fourier transform analysis equation (i.e., $X(\Omega) = \sum_{-\infty}^{\infty} x(n)e^{-j\Omega n}$) converges in the ***MSE sense*** if

$$\sum_{k=-\infty}^{\infty} |x(k)|^2 < \infty$$

(i.e., x is ***square summable***).

- For a bounded Fourier transform X, the Fourier transform synthesis equation (i.e., $x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$) will always converge, since the integration interval is finite.

Section 11.3

Properties of the Fourier Transform

Properties of the (DT) Fourier Transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Translation	$x(n-n_0)$	$e^{-j\Omega n_0}X(\Omega)$
Modulation	$e^{j\Omega_0 n}x(n)$	$X(\Omega-\Omega_0)$
Conjugation	$x^*(n)$	$X^*(-\Omega)$
Time Reversal	$x(-n)$	$X(-\Omega)$
Upsampling	$(\uparrow M)x(n)$	$X(M\Omega)$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1}X\left(\frac{\Omega-2\pi k}{M}\right)$
Convolution	$x_1 * x_2(n)$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{2\pi}X_1(\theta)X_2(\Omega-\theta)d\theta$
Freq.-Domain Diff.	$nx(n)$	$j\frac{d}{d\Omega}X(\Omega)$
Differencing	$x(n)-x(n-1)$	$\left(1-e^{-j\Omega}\right)X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n}x(k)$	$\frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega)+\pi X(0)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$

Property	
Periodicity	$X(\Omega) = X(\Omega + 2\pi)$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} \lvert x(n) \rvert^2 = \frac{1}{2\pi} \int_{2\pi} \lvert X(\Omega) \rvert^2 d\Omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

(DT) Fourier Transform Pairs

Pair	$x(n)$	$X(\Omega)$
1	$\delta(n)$	1
2	1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
3	$u(n)$	$\frac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$
4	$a^n u(n), \lvert a\rvert < 1$	$\frac{e^{j\Omega}}{e^{j\Omega}-a}$
5	$-a^n u(-n-1), \lvert a\rvert > 1$	$\frac{e^{j\Omega}}{e^{j\Omega}-a}$
6	$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\frac{1-a^2}{1-2a\cos\Omega+a^2}$
7	$\cos\Omega_0 n$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)]$
8	$\sin\Omega_0 n$	$j\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega + \Omega_0 - 2\pi k) - \delta(\Omega - \Omega_0 - 2\pi k)]$
9	$(\cos\Omega_0 n)u(n)$	$\frac{e^{j2\Omega}-e^{j\Omega}\cos\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1} + \frac{\pi}{2} \sum_{k=-\infty}^{\infty} [\delta(\Omega - 2\pi k - \Omega_0) + \delta(\Omega - 2\pi k + \Omega_0)]$
10	$(\sin\Omega_0 n)u(n)$	$\frac{e^{j\Omega}\sin\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1} + \frac{\pi}{2j} \sum_{k=-\infty}^{\infty} [\delta(\Omega - 2\pi k - \Omega_0) - \delta(\Omega - 2\pi k + \Omega_0)]$
11	$\frac{B}{\pi} \operatorname{sinc} Bn, 0 < B < \pi$	$\sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\frac{\Omega - 2\pi k}{2B}\right)$
12	$u(n) - u(n-M)$	$e^{-j\Omega(M-1)/2} \left(\frac{\sin(M\Omega/2)}{\sin(\Omega/2)}\right)$

Periodicity

- Recall the definition of the Fourier transform X of the sequence x:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

- For all integer k, we have that

$$\begin{aligned} X(\Omega+2\pi k) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\Omega+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\Omega n+2\pi kn)} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\ &= X(\Omega). \end{aligned}$$

- Thus, the Fourier transform X of the sequence x is always ***2π-periodic***.

Linearity

- If $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ and $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$, then

$$a_1x_1(n) + a_2x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} a_1X_1(\Omega) + a_2X_2(\Omega),$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Fourier transform.

- If $x(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$x(n-n_0) \stackrel{\text{DTFT}}{\longleftrightarrow} e^{-j\Omega n_0} X(\Omega),$$

where n_0 is an arbitrary integer.

- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

Modulation

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$e^{j\Omega_0 n}x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega - \Omega_0),$$

  where Ω_0 is an arbitrary real constant.

- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then
$$x^*(n) \overset{\text{DTFT}}{\longleftrightarrow} X^*(-\Omega).$$
- This is known as the **conjugation property** of the Fourier transform.

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$x(-n) \overset{\text{DTFT}}{\longleftrightarrow} X(-\Omega).$$

- This is known as the **time-reversal property** of the Fourier transform.

Upsampling

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$(\uparrow M)x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(M\Omega).$$

- This is known as the **upsampling property** of the Fourier transform.

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$(\downarrow M)x(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\Omega-2\pi k}{M}\right).$$

- This is known as the **downsampling property** of the Fourier transform.

Convolution

- If $x_1(n) \xleftrightarrow{\text{DTFT}} X_1(\Omega)$ and $x_2(n) \xleftrightarrow{\text{DTFT}} X_2(\Omega)$, then

$$x_1 * x_2(n) \xleftrightarrow{\text{DTFT}} X_1(\Omega)X_2(\Omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

- If $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ and $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$, then

$$x_1(n)x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} X_1(\theta)X_2(\Omega-\theta)d\theta.$$

- This is known as the **multiplication (or time-domain multiplication) property** of the Fourier transform.
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$nx(n) \overset{\text{DTFT}}{\longleftrightarrow} j\tfrac{d}{d\Omega}X(\Omega).$$

- This is known as the **frequency-domain differentiation property** of the Fourier transform.

Differencing

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then
$$x(n) - x(n-1) \overset{\text{DTFT}}{\longleftrightarrow} \left(1 - e^{-j\Omega}\right) X(\Omega).$$
- This is known as the **differencing property** of the Fourier transform.
- Note that this property follows quite trivially from the linearity and translation properties of the Fourier transform.

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$\sum_{k=-\infty}^{n} x(k) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-1} X(\Omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).$$

- This is known as the **accumulation property** of the Fourier transform.

- If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega$$

(i.e., the energy of x and energy of X are equal up to a factor of 2π).

- This is known as **Parseval's relation**.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform ***preserves energy*** (up to a scale factor).

Even and Odd Symmetry

- For a sequence x with Fourier transform X, the following assertions hold:
  1. x is even $\Leftrightarrow X$ is even; and
  2. x is odd $\Leftrightarrow X$ is odd.
- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

- A sequence x is ***real*** if and only if its Fourier transform X satisfies

$$X(\Omega) = X^*(-\Omega) \text{ for all } \Omega$$

(i.e., X is ***conjugate symmetric***).

- Thus, for a real-valued sequence, the portion of the graph of a Fourier transform for negative values of frequency Ω is ***redundant***, as it is completely determined by symmetry.
- From properties of complex numbers, one can show that $X(\Omega) = X^*(-\Omega)$ is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \arg X(\Omega) = -\arg X(-\Omega)$$

(i.e., $|X(\Omega)|$ is ***even*** and $\arg X(\Omega)$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that X is real.

- The DTFT analysis and synthesis equations are, respectively, given by

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x(k)e^{-jk\Omega} \quad \text{and} \quad x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{jn\Omega}d\Omega.$$

- The CTFS synthesis and analysis equations are, respectively, given by

$$x_{\mathsf{c}}(t) = \sum_{k=-\infty}^{\infty} a(k)e^{jk(2\pi/T)t} \quad \text{and} \quad a(n) = \frac{1}{T}\int_{T} x_{\mathsf{c}}(t)e^{-jn(2\pi/T)t}dt,$$

which can be rewritten, respectively, as

$$x_{\mathsf{c}}(t) = \sum_{k=-\infty}^{\infty} a(-k)e^{-jk(2\pi/T)t} \quad \text{and} \quad a(-n) = \frac{1}{T}\int_{T} x_{\mathsf{c}}(t)e^{jn(2\pi/T)t}dt.$$

- The CTFS synthesis equation with $T = 2\pi$ corresponds to the DTFT analysis equation with $X = x_{\mathsf{c}}$, $\Omega = t$, and $x(n) = a(-n)$.
- The CTFS analysis equation with $T = 2\pi$ corresponds to the DTFT synthesis equation with $X = x_{\mathsf{c}}$ and $x(n) = a(-n)$.
- Consequently, the DTFT X of the sequence x can be viewed as a CTFS representation of the 2π-periodic spectrum X.

Fourier Transform of Periodic Sequences

- The Fourier transform can be generalized to also handle periodic sequences.
- Consider an N-periodic sequence x.
- Define the sequence x_N as

$$x_N(n) = \begin{cases} x(n) & 0 \leq n < N \\ 0 & \text{otherwise.} \end{cases}$$

(i.e., $x_N(n)$ is equal to $x(n)$ over a single period and zero elsewhere).

- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_N denote the Fourier transforms of x and x_N, respectively.
- The following relationships can be shown to hold:

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_N\left(\frac{2\pi k}{N}\right) \delta\left(\Omega - \frac{2\pi k}{N}\right),$$

$$a_k = \frac{1}{N} X_N\left(\frac{2\pi k}{N}\right), \quad \text{and} \quad X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right).$$

- The Fourier series coefficient sequence a is produced by sampling X_N at integer multiples of the fundamental frequency $\frac{2\pi}{N}$ and scaling the resulting sequence by $\frac{1}{N}$.
- The Fourier transform of a periodic sequence can only be nonzero at integer multiples of the fundamental frequency.

Section 11.4

Fourier Transform and Frequency Spectra of Sequences

Frequency Spectra of Sequences

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on sequences.
- That is, instead of viewing a sequence as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a sequence as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- The Fourier transform X of a sequence x provides a means to ***quantify*** how much information x has at different frequencies.
- The distribution of information in a sequence over different frequencies is referred to as the ***frequency spectrum*** of the sequence.

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\Omega)$ expressed in ***polar form*** as follows:

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|\, e^{j[\Omega n + \arg X(\Omega)]}d\Omega.$$

- In effect, the quantity $|X(\Omega)|$ is a ***weight*** that determines how much the complex sinusoid at frequency Ω contributes to the integration result $x(n)$.
- Perhaps, this can be more easily seen if we express the above integral as the ***limit of a sum***, derived from an approximation of the integral using the area of rectangles, as shown on the next slide. [Recall that $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$.]

- Expressing the integral (from the previous slide) as the ***limit of a sum***, we obtain

$$x(n) = \lim_{\ell \to \infty} \frac{1}{2\pi} \sum_{k=1}^{\ell} \Delta\Omega \left|X(\Omega')\right| e^{j[\Omega' n + \arg X(\Omega')]},$$

  where $\Delta\Omega = \frac{2\pi}{\ell}$ and $\Omega' = k\Delta\Omega$.
- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\Omega' = k\Delta\Omega$ that has had its ***amplitude scaled*** by a factor of $|X(\Omega')|$ and has been ***time shifted*** by an amount that depends on $\arg X(\Omega')$.
- For a given $\Omega' = k\Delta\Omega$ (which is associated with the kth term in the summation), the larger $|X(\Omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\Omega' n}$ will be, and therefore the larger the contribution the kth term will make to the overall summation.
- In this way, we can use $|X(\Omega')|$ as a ***measure*** of how much information a sequence x has at the frequency Ω'.

- The Fourier transform X of the sequence x is referred to as the **frequency spectrum** of x.
- The magnitude $|X(\Omega)|$ of the Fourier transform X is referred to as the **magnitude spectrum** of x.
- The argument $\arg X(\Omega)$ of the Fourier transform X is referred to as the **phase spectrum** of x.
- Since the Fourier transform is a function of a real variable, a sequence can potentially have information at any real frequency.
- Earlier, we saw that for periodic sequences, the Fourier transform can only be nonzero at integer multiples of the fundamental frequency.
- So, the Fourier transform and Fourier series give a consistent picture in terms of frequency spectra.
- Since the frequency spectrum is complex (in the general case), it is ***usually represented using two plots***, one showing the magnitude spectrum and one showing the phase spectrum.

- Recall that, for a ***real*** sequence x, the Fourier transform X of x satisfies

$$X(\Omega) = X^*(-\Omega)$$

(i.e., X is ***conjugate symmetric***), which is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \arg X(\Omega) = -\arg X(-\Omega).$$

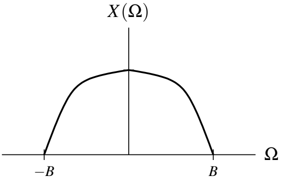
- Since $|X(\Omega)| = |X(-\Omega)|$, the magnitude spectrum of a real sequence is always ***even***.
- Similarly, since $\arg X(\Omega) = -\arg X(-\Omega)$, the phase spectrum of a real sequence is always ***odd***.
- Due to the symmetry in the frequency spectra of real sequences, we typically ***ignore negative frequencies*** when dealing with such sequences.
- In the case of sequences that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

Bandwidth

- A sequence x with Fourier transform X is said to be **bandlimited** if, for some nonnegative real constant B,

$$X(\Omega) = 0 \text{ for all } \Omega \text{ satisfying } |\Omega| > B.$$

- In the context of real sequences, we usually refer to B as the **bandwidth** of the sequence x.
- The (real) sequence with the Fourier transform X shown below has bandwidth B.



- One can show that a sequence ***cannot be both time limited and bandlimited***. (This follows from the time/frequency scaling property of the Fourier transform.)

Section 11.5

Fourier Transform and LTI Systems

Frequency Response of LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(n) = x * h(n)$, we have that

$$Y(\Omega) = X(\Omega)H(\Omega).$$

- The function H is called the **frequency response** of the system.
- A LTI system is ***completely characterized*** by its frequency response H.
- The above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output signals.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\Omega)$ in terms of its magnitude $|H(\Omega)|$ and argument $\arg H(\Omega)$.
- The quantity $|H(\Omega)|$ is called the **magnitude response** of the system.
- The quantity $\arg H(\Omega)$ is called the **phase response** of the system.
- Since $Y(\Omega) = X(\Omega)H(\Omega)$, we trivially have that

$$|Y(\Omega)| = |X(\Omega)|\,|H(\Omega)| \quad \text{and} \quad \arg Y(\Omega) = \arg X(\Omega) + \arg H(\Omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.

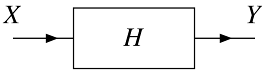
- Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is ***real***, then

$$|H(\Omega)| = |H(-\Omega)| \quad \text{and} \quad \arg H(\Omega) = -\arg H(-\Omega)$$

(i.e., the magnitude response $|H(\Omega)|$ is ***even*** and the phase response $\arg H(\Omega)$ is ***odd***).

Block Diagram Representations of LTI Systems

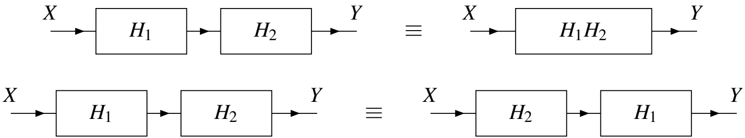
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the frequency domain as shown below.



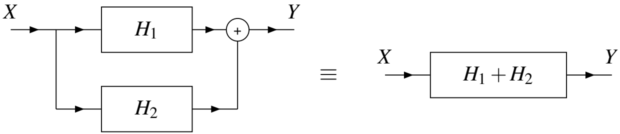
- Since a LTI system is completely characterized by its frequency response, we typically label the system with this quantity.

Interconnection of LTI Systems

- The ***series*** interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with frequency response H_1H_2. That is, we have the equivalences shown below.



- The ***parallel*** interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with the frequency response $H_1 + H_2$. That is, we have the equivalence shown below.



- Many LTI systems of practical interest can be represented using an ***Nth-order linear difference equation with constant coefficients***.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k).$$

- Let h denote the impulse response of the system, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- One can show that $H(\Omega)$ is given by

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} a_k (e^{j\Omega})^{-k}}{\sum_{k=0}^{N} b_k (e^{j\Omega})^{-k}} = \frac{\sum_{k=0}^{M} a_k e^{-jk\Omega}}{\sum_{k=0}^{N} b_k e^{-jk\Omega}}.$$

- Each of the numerator and denominator of H is a ***polynomial*** in $e^{-j\Omega}$.
- Thus, H is a ***rational function*** in the variable $e^{-j\Omega}$.

Section 11.6

Application: Filtering

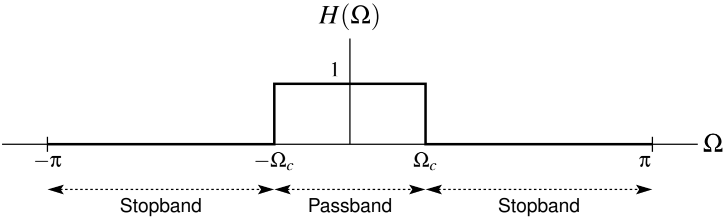
Filtering

- In many applications, we want to ***modify the spectrum*** of a signal by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a signal is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

- An **ideal lowpass filter** eliminates all baseband frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining baseband frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\Omega) = \begin{cases} 1 & |\Omega| \leq \Omega_c \\ 0 & \Omega_c < |\Omega| \leq \pi, \end{cases}$$

  where Ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

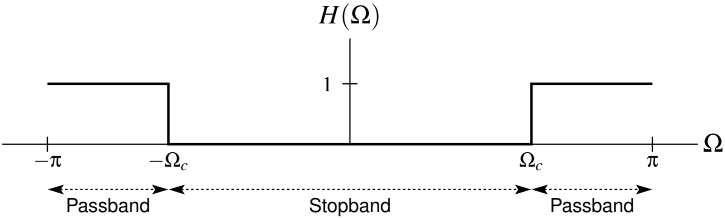


Ideal Highpass Filter

- An **ideal highpass filter** eliminates all baseband frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining baseband frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\Omega) = \begin{cases} 1 & \Omega_c < |\Omega| \leq \pi \\ 0 & |\Omega| \leq \Omega_c, \end{cases}$$

  where Ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

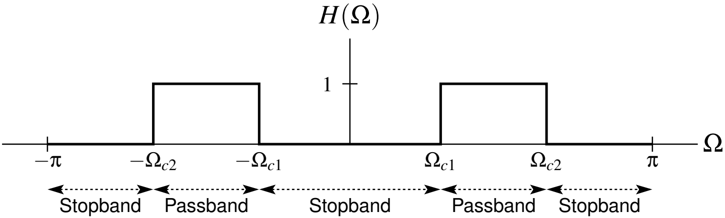


Ideal Bandpass Filter

- An **ideal bandpass filter** eliminates all baseband frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining baseband frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\Omega) = \begin{cases} 1 & \Omega_{c1} \leq |\Omega| \leq \Omega_{c2} \\ 0 & |\Omega| < \Omega_{c1} \text{ or } \Omega_{c2} < |\Omega| < \pi, \end{cases}$$

  where the limits of the passband are Ω_{c1} and Ω_{c2}.
- A plot of this frequency response is given below.



Part 12

Z Transform (ZT)

- Another important mathematical tool in the study of signals and systems is known as the z transform.
- The z transform can be viewed as a ***generalization of the Fourier transform***.
- Due to its more general nature, the z transform has a number of ***advantages*** over the Fourier transform.
- First, the z transform representation exists for some sequences that do not have Fourier transform representations. So, we can handle a ***larger class of sequences*** with the z transform.
- Second, since the z transform is a more general tool, it can provide ***additional insights*** beyond those facilitated by the Fourier transform.

Motivation Behind the Z Transform (Continued)

- Earlier, we saw that complex exponentials are eigensequences of LTI systems.
- In particular, for a LTI system $\mathcal{H}$ with impulse response h, we have that

$$\mathcal{H}\{z^n\}(n) = H(z)z^n \quad \text{where} \quad H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}.$$

- Previously, we referred to H as the system function.
- As it turns out, H is the z transform of h.
- Since the z transform has already appeared earlier in the context of LTI systems, it is clearly a useful tool.
- Furthermore, as we will see, the z transform has many additional uses.

Section 12.1

Z Transform

(Bilateral) Z Transform

- The (bilateral) **z transform** of the sequence x, denoted $\mathcal{Z}x$ or X, is defined as

$$\mathcal{Z}x(z) = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

- The **inverse z transform** of X, denoted $\mathcal{Z}^{-1}X$ or x, is then given by

$$\mathcal{Z}^{-1}X(n) = x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} dz,$$

  where Γ is a counterclockwise closed circular contour centered at the origin and with radius r such that Γ is in the ROC of X.
- We refer to x and X as a **z transform pair** and denote this relationship as

$$x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z).$$

- In practice, we do not usually compute the inverse z transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

Bilateral and Unilateral Z Transform

- Two different versions of the z transform are commonly used:
  1. the ***bilateral*** (or ***two-sided***) z transform; and
  2. the ***unilateral*** (or ***one-sided***) z transform.
- The unilateral z transform is most frequently used to solve systems of linear difference equations with nonzero initial conditions.
- As it turns out, the only difference between the definitions of the bilateral and unilateral z transforms is in the ***lower limit of summation***.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0.
- For the most part, we will focus our attention primarily on the bilateral z transform.
- We will, however, briefly introduce the unilateral z transform as a tool for solving difference equations.
- Unless otherwise noted, all subsequent references to the z transform should be understood to mean ***bilateral*** z transform.

Relationship Between Z and Fourier Transforms

- Let X and X_F denote the z and (DT) Fourier transforms of x, respectively.
- The function $X(z)$ evaluated at $z = e^{j\Omega}$ (where Ω is real) yields $X_\mathsf{F}(\Omega)$. That is,

$$X(e^{j\Omega}) = X_\mathsf{F}(\Omega).$$

- Due to the preceding relationship, the Fourier transform of x is sometimes written as $X(e^{j\Omega})$.
- The function $X(z)$ evaluated at an arbitrary complex value $z = re^{j\Omega}$ (where $r = |z|$ and $\Omega = \arg z$) can also be expressed in terms of a Fourier transform involving x. In particular, we have

$$X(re^{j\Omega}) = X'_\mathsf{F}(\Omega),$$

  where X'_F is the (DT) Fourier transform of $x'(n) = r^{-n}x(n)$.
- So, in general, the z transform of x is the Fourier transform of an exponentially-weighted version of x.
- Due to this weighting, the z transform of a sequence may exist when the Fourier transform of the same sequence does not.

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Section 12.2

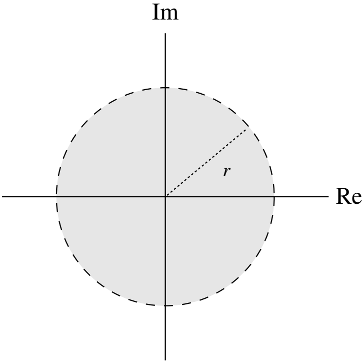
Region of Convergence (ROC)

Disc

- A **disc** with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| < r,$$

where r is a real constant and $r > 0$.

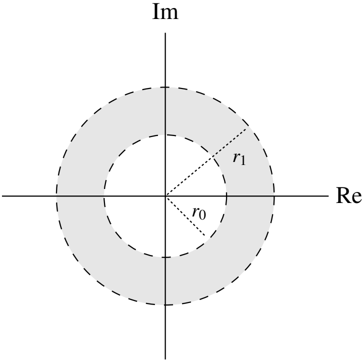


Annulus

- An **annulus** with center 0, inner radius r_0, and outer radius r_1 is the set of all complex numbers z satisfying

$$r_0 < |z| < r_1,$$

where r_0 and r_1 are real constants and $0 < r_0 < r_1$.

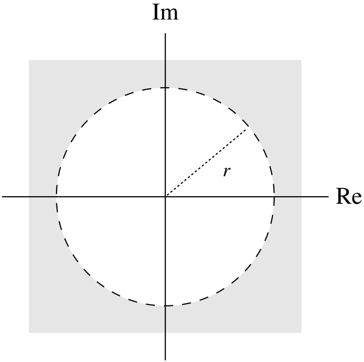


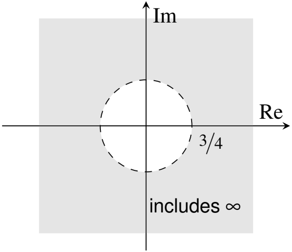
Circle Exterior

- The **exterior of a circle** with center 0 and radius r is the set of all complex numbers z satisfying

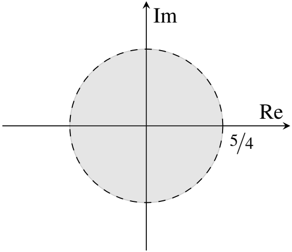
$$|z| > r,$$

where r is a real constant and $r > 0$.

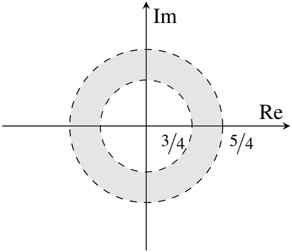




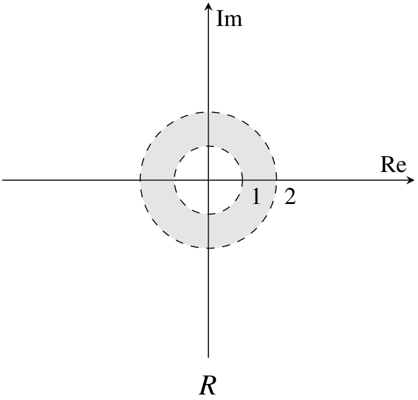
R_1

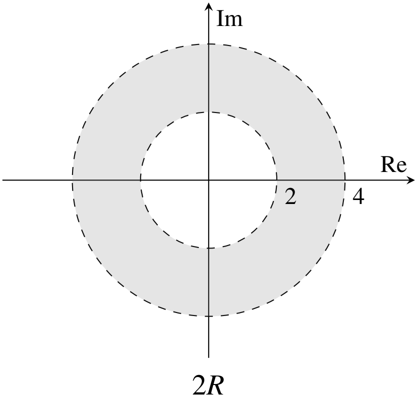


R_2

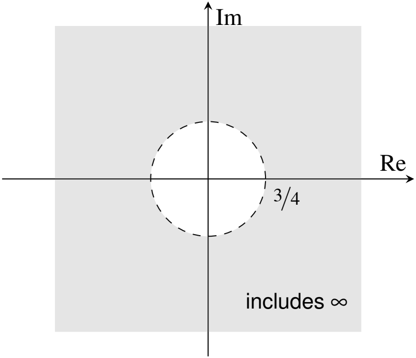


$R_1 \cap R_2$

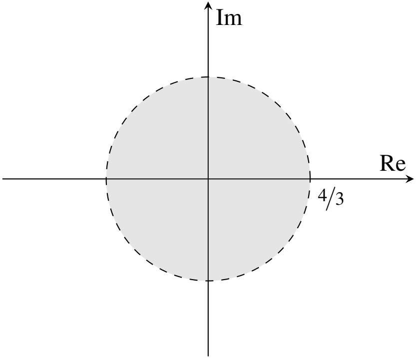

Im
Re
1
2
R


Im
Re
2
4
2R

Example: Reciprocal of a Set



R



R^{-1}

Region of Convergence (ROC)

- As we saw earlier, for a sequence x, the complete specification of its z transform X requires not only an algebraic expression for X, but also the ROC associated with X.
- Two very different sequences can have the same algebraic expressions for X.
- Now, we examine some of the constraints on the ROC (of the z transform) for various classes of sequences.

Properties of the ROC

1. The ROC consists of ***concentric circles centered at*** 0 in the complex plane.
2. If the sequence x has a ***rational*** z transform, then the ROC ***does not contain any poles***, and the ROC is ***bounded by poles or extends to*** ∞.
3. If the sequence x is ***finite duration***, then the ROC is the ***entire complex plane***, except possibly 0 and/or ∞.
4. If the sequence x is ***right sided but not left sided*** and the circle $|z| = r_0$ is in the ROC, then all (finite) values of z for which $|z| > r_0$ will also be in the ROC (i.e., the ROC is the ***exterior of a circle***, possibly excluding ∞).
5. If the sequence x is ***left sided but not right sided*** and the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC (i.e., the ROC is a ***disc***, possibly excluding 0).
6. If the sequence x is ***two sided*** and the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring that includes this circle (i.e., the ROC is an ***annulus***).

7. If the z transform X of x is ***rational*** and x is ***right sided***, then the ROC is the region outside the circle of radius equal to the largest magnitude of the poles of X (i.e., ***outside the outermost pole***).
8. If the z transform X of x is ***rational*** and x is ***left sided***, then the ROC is the region inside the circle of radius equal to the smallest magnitude of the nonzero poles of X and extending inward to, and possibly including, 0 (i.e., ***inside the innermost nonzero pole***).

- Some of the preceding properties are redundant (e.g., properties 1, 2, and 4 imply property 7).
- The ROC must always be of the form of one of the following:
  1. a disc centered at 0, possibly excluding the origin
  2. an annulus centered 0
  3. the exterior of a circle centered at 0, possibly excluding ∞
  4. the entire complex plane, possibly excluding 0 and/or ∞

Section 12.3

Properties of the Z Transform

Properties of the Z Transform

Property	Time Domain	Z Domain	ROC
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$	At least $R_1 \cap R_2$
Translation	$x(n-n_0)$	$z^{-n_0}X(z)$	R except possible addition/deletion of 0
Modulation	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert R$
Time Reversal	$x(-n)$	$X(1/z)$	R^{-1}
Upsampling	$(\uparrow M)x(n)$	$X(z^M)$	$R^{1/M}$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}z^{1/M}\right)$	R^M
Conjugation	$x^*(n)$	$X^*(z^*)$	R
Convolution	$x_1 * x_2(n)$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$
Z-Domain Diff.	$nx(n)$	$-z\frac{d}{dz}X(z)$	R
Differencing	$x(n)-x(n-1)$	$(1-z^{-1})X(z)$	At least $R \cap \lvert z\rvert > 0$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{z}{z-1}X(z)$	At least $R \cap \lvert z\rvert > 1$

Property	
Initial Value Theorem	$x(0) = \lim_{z\to\infty} X(z)$
Final Value Theorem	$\lim_{n\to\infty} x(n) = \lim_{z\to 1}[(z-1)X(z)]$

Z Transform Pairs

Pair	$x(n)$	$X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{z}{z-1}$	$\|z\| > 1$
3	$-u(-n-1)$	$\frac{z}{z-1}$	$\|z\| < 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
5	$-nu(-n-1)$	$\frac{z}{(z-1)^2}$	$\|z\| < 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
7	$-a^n u(-n-1)$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
9	$-na^n u(-n-1)$	$\frac{az}{(z-a)^2}$	$\|z\| < \|a\|$
10	$(\cos\Omega_0 n)u(n)$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\|z\| > 1$
11	$(\sin\Omega_0 n)u(n)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\|z\| > 1$
12	$(a^n\cos\Omega_0 n)u(n)$	$\frac{z(z-a\cos\Omega_0)}{z^2-2az\cos\Omega_0+a^2}$	$\|z\| > \|a\|$
13	$(a^n\sin\Omega_0 n)u(n)$	$\frac{az\sin\Omega_0}{z^2-2az\cos\Omega_0+a^2}$	$\|z\| > \|a\|$

Linearity

- If $x_1(n) \xleftrightarrow{\text{ZT}} X_1(z)$ with ROC R_1 and $x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$ with ROC R_2, then

  $$a_1x_1(n) + a_2x_2(n) \xleftrightarrow{\text{ZT}} a_1X_1(z) + a_2X_2(z) \quad \text{with ROC } R \text{ containing } R_1 \cap R_2,$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the z transform.
- The ROC always contains the intersection but could be larger (in the case that pole-zero cancellation occurs).

Translation (Time Shifting)

- If $x(n) \stackrel{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$x(n-n_0) \stackrel{\text{ZT}}{\longleftrightarrow} z^{-n_0}X(z) \text{ with ROC } R',$$

  where n_0 is an integer constant and R' is the same as R except for the possible addition or deletion of zero or infinity.
- This is known as the **translation (or time-shifting) property** of the z transform.

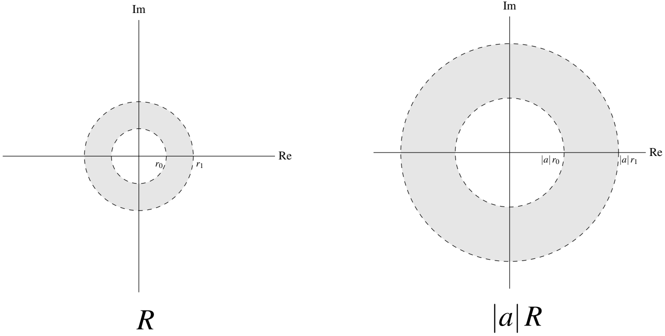
Z-Domain Scaling

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$a^n x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z/a) \quad \text{with ROC } |a|R,$$

where a is a nonzero constant.

- This is known as the **z-domain scaling property** of the z transform.
- As illustrated below, the ROC R is ***scaled*** by $|a|$.

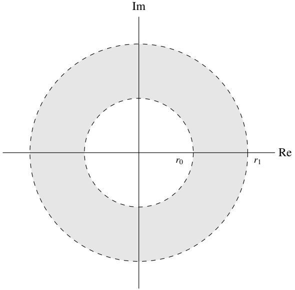


Time Reversal

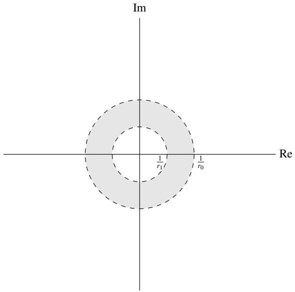
- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$x(-n) \overset{\text{ZT}}{\longleftrightarrow} X(1/z) \quad \text{with ROC } 1/R.$$

- This is known as the **time-reversal property** of the z transform.
- As illustrated below, the ROC R is ***reciprocated***.



R



$1/R$

Upsampling

- Define $(\uparrow M)x(n)$ as

$$(\uparrow M)x(n) = \begin{cases} x(n/M) & n/M \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

- If $x(n) \xleftrightarrow{\text{ZT}} X(z)$ with ROC R, then

$$(\uparrow M)x(n) \xleftrightarrow{\text{ZT}} X(z^M) \quad \text{with ROC } R^{1/M}.$$

- This is known as the **upsampling (or time-expansion) property** of the z transform.

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$(\downarrow M)x(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M} z^{1/M}\right) \quad \text{with ROC } R^M.$$

- This is known as the **downsampling property** of the z transform.

Conjugation

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$x^*(n) \overset{\text{ZT}}{\longleftrightarrow} X^*(z^*) \quad \text{with ROC } R.$$

- This is known as the **conjugation property** of the z transform.

Convolution

- If $x_1(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_1(z)$ with ROC R_1 and $x_2(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_2(z)$ with ROC R_2, then

$$x_1 * x_2(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_1(z)X_2(z) \quad \text{with ROC containing } R_1 \cap R_2.$$

- This is known that the **convolution (or time-domain convolution) property** of the z transform.
- The ROC always contains the intersection but can be larger than the intersection (if pole-zero cancellation occurs).
- Convolution in the time domain becomes ***multiplication*** in the z domain.
- This can make dealing with LTI systems much easier in the z domain than in the time domain.

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$nx(n) \overset{\text{ZT}}{\longleftrightarrow} -z\frac{d}{dz}X(z) \quad \text{with ROC } R.$$

- This is known as the **z-domain differentiation property** of the z transform.

Differencing

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$x(n) - x(n-1) \overset{\text{ZT}}{\longleftrightarrow} (1 - z^{-1})X(z) \text{ for ROC containing } R \cap |z| > 0.$$

- This is known as the **differencing property** of the z transform.
- Differencing in the time domain becomes multiplication by $1 - z^{-1}$ in the z domain.
- This can make dealing with difference equations much easier in the z domain than in the time domain.

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$\sum_{k=-\infty}^{n} x(k) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1}X(z) \text{ for ROC containing } R \cap |z| > 1.$$

- This is known as the **accumulation property** of the z transform.

Initial Value Theorem

- For a sequence x with z transform X, if x is causal, then

$$x(0) = \lim_{z \to \infty} X(z).$$

- This result is known as the **initial-value theorem**.

- For a sequence x with z transform X, if x is causal and $\lim_{n\to\infty} x(n)$ exists, then

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}[(z-1)X(z)].$$

- This result is known as the **final-value theorem**.

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Section 12.4

Determination of Inverse Z Transform

Finding the Inverse Z Transform

- Recall that the inverse z transform x of X is given by

$$x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} dz,$$

  where Γ is a counterclockwise closed circular contour centered at the origin and with radius r such that Γ is in the ROC of X.
- Unfortunately, the above contour integration can often be ***quite tedious*** to compute.
- Consequently, we do not usually compute the inverse z transform directly using the above equation.
- For rational functions, the inverse z transform can be more easily computed using ***partial fraction expansions***.
- Using a partial fraction expansion, we can express a rational function as a sum of lower-order rational functions whose inverse z transforms can typically be found in tables.

Section 12.5

Z Transform and LTI Systems

System Function of LTI Systems

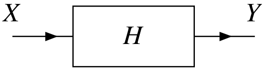
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the z transforms of x, y, and h, respectively.
- Since $y(n) = x * h(n)$, the system is characterized in the z domain by

$$Y(z) = X(z)H(z).$$

- As a matter of terminology, we refer to H as the **system function** (or **transfer function**) of the system (i.e., the system function is the z transform of the impulse response).
- When viewed in the z domain, a LTI system forms its output by multiplying its input with its system function.
- A LTI system is completely characterized by its system function H.
- If the ROC of H includes the unit circle $|z| = 1$, then $H(e^{j\Omega})$ is the ***frequency response*** of the LTI system.

Block Diagram Representation of LTI Systems

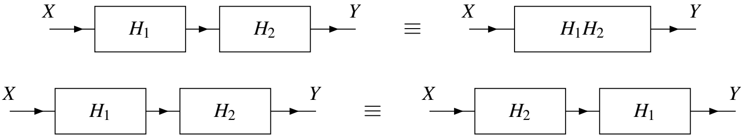
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the z transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the z domain as shown below.



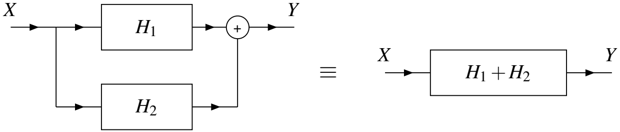
- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

Interconnection of LTI Systems

- The ***series*** interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with system function $H = H_1 H_2$. That is, we have the equivalences shown below.



- The ***parallel*** interconnection of the LTI systems with impulse responses H_1 and H_2 is a LTI system with the system function $H = H_1 + H_2$. That is, we have the equivalence shown below.



- If a LTI system is ***causal***, its impulse response is causal, and therefore ***right sided***. From this, we have the result below.
- **Theorem.** A LTI system is ***causal*** if and only if the ROC of the system function ***includes*** ∞, which corresponds to the ROC being: 1) the exterior of a circle including ∞, or 2) the entire complex plane, including ∞ and possibly excluding 0.
- **Theorem.** A LTI system with a ***rational*** system function H is causal if and only if
  1. the ROC is the exterior of a circle ***outside the outermost pole***; and
  2. with $H(z)$ expressed as a ratio of polynomials in z the order of the numerator polynomial ***does not exceed*** the order of the denominator polynomial.

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is ***BIBO stable*** if and only if the ROC of its system function includes the (entire) ***unit circle*** (i.e., $|z| = 1$).
- **Theorem.** A ***causal*** LTI system with a ***rational*** system function H is BIBO stable if and only if all of the poles of H lie inside the unit circle (i.e., each of the poles has a ***magnitude less than one***).

Invertibility

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(z)H_{\text{inv}}(z) = 1,$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(z) = \frac{1}{H(z)}.$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is ***not necessarily unique***.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in ***one specific choice*** of inverse system (due to these additional constraints of stability and/or causality).

LTI Systems and Difference Equations

- Many LTI systems of practical interest can be represented using an ***Nth-order linear difference equation with constant coefficients***.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k) \quad \text{where} \quad M \leq N.$$

- Let h denote the impulse response of the system, and let X, Y, and H denote the z transforms of x, y, and h, respectively.
- One can show that $H(z)$ is given by

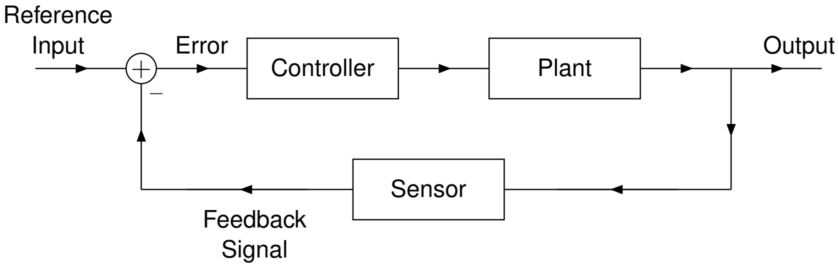
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}.$$

- Observe that, for a system of the form considered above, the system function is always ***rational***.

Section 12.6

Application: Analysis of Control Systems

Feedback Control Systems



- **input**: ***desired value*** of the quantity to be controlled
- **output**: ***actual value*** of the quantity to be controlled
- **error**: ***difference*** between the desired and actual values
- **plant**: system to be controlled
- **sensor**: device used to measure the actual output
- **controller**: device that monitors the error and changes the input of the plant with the goal of forcing the error to zero

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the z domain than in the time domain.
- Therefore, the z domain is extremely useful for the stability analysis of systems.

Section 12.7

Unilateral Z Transform

- The **unilateral z transform** of the sequence x, denoted $\mathcal{Z}_{\mathsf{u}}x$ or X, is defined as

$$\mathcal{Z}_{\mathsf{u}}x(z) = X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}.$$

- The unilateral z transform is related to the bilateral z transform as follows:

$$\mathcal{Z}_{\mathsf{u}}x(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(n)u(n)z^{-n} = \mathcal{Z}\{xu\}(z).$$

- In other words, the unilateral z transform of the sequence x is simply the bilateral z transform of the sequence xu.
- Since $\mathcal{Z}_{\mathsf{u}}x = \mathcal{Z}\{xu\}$ and xu is always a ***right-sided*** sequence, the ROC associated with $\mathcal{Z}_{\mathsf{u}}x$ is always the ***exterior of a circle***.
- For this reason, we often ***do not explicitly indicate the ROC*** when working with the unilateral z transform.

- With the unilateral z transform, the same inverse transform equation is used as in the bilateral case.
- The unilateral z transform is ***only invertible for causal sequences***. In particular, we have

$$\begin{aligned}\mathcal{Z}_{\mathrm{u}}^{-1}\{\mathcal{Z}_{\mathrm{u}}\{x\}\}(n) &= \mathcal{Z}_{\mathrm{u}}^{-1}\{\mathcal{Z}\{xu\}\}(n) \\ &= \mathcal{Z}^{-1}\{\mathcal{Z}\{xu\}\}(n) \\ &= x(n)u(n) \\ &= \begin{cases} x(n) & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

- For a noncausal sequence x, we can only recover $x(n)$ for $n \geq 0$.

- Due to the close relationship between the unilateral and bilateral z transforms, these two transforms have some similarities in their properties.
- Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways.

Properties of the Unilateral Z Transform

Property	Time Domain	Z Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$
Time Delay	$x(n-1)$	$z^{-1}X(z)+x(-1)$
Time Advance	$x(n+1)$	$zX(z)-zx(0)$
Modulation	$a^nx(n)$	$X(a^{-1}z)$
	$e^{j\Omega_0 n}x(n)$	$X(e^{-j\Omega_0}z)$
Upsampling	$(\uparrow M)x(n)$	$X(z^M)$
Conjugation	$x^*(n)$	$X^*(z^*)$
Convolution	$x_1 * x_2(n)$, x_1 and x_2 are causal	$X_1(z)X_2(z)$
Z-Domain Diff.	$nx(n)$	$-z\frac{d}{dz}X(z)$
Differencing	$x(n)-x(n-1)$	$(1-z^{-1})X(z)-x(-1)$
Accumulation	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$

Property	
Initial Value Theorem	$x(0)=\lim_{z\to\infty} X(z)$
Final Value Theorem	$\lim_{n\to\infty} x(n)=\lim_{z\to 1}[(z-1)X(z)]$

Pair	$x(n),\ n \geq 0$	$X(z)$
1	$\delta(n)$	1
2	1	$\frac{z}{z-1}$
3	n	$\frac{z}{(z-1)^2}$
4	a^n	$\frac{z}{z-a}$
5	$a^n n$	$\frac{az}{(z-a)^2}$
6	$\cos \Omega_0 n$	$\frac{z(z-\cos \Omega_0)}{z^2-2(\cos \Omega_0)z+1}$
7	$\sin \Omega_0 n$	$\frac{z \sin \Omega_0}{z^2-2(\cos \Omega_0)z+1}$
8	$\lvert a\rvert^n \cos \Omega_0 n$	$\frac{z(z-\lvert a\rvert \cos \Omega_0)}{z^2-2\lvert a\rvert(\cos \Omega_0)z+\lvert a\rvert^2}$
9	$\lvert a\rvert^n \sin \Omega_0 n$	$\frac{z\lvert a\rvert \sin \Omega_0}{z^2-2\lvert a\rvert(\cos \Omega_0)z+\lvert a\rvert^2}$

- Many systems of interest in engineering applications can be characterized by constant-coefficient linear difference equations.
- One common use of the unilateral z transform is in solving constant-coefficient linear difference equations with nonzero initial conditions.

Part 13

Complex Analysis

- A **complex number** is a number of the form $z = x + jy$ where x and y are real numbers and j is the constant defined by $j^2 = -1$ (i.e., $j = \sqrt{-1}$).
- The **Cartesian form** of the complex number z expresses z in the form

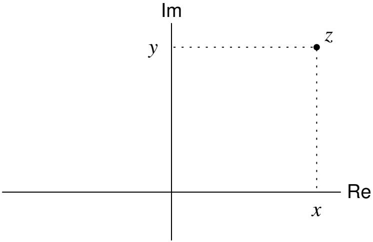
$$z = x + jy,$$

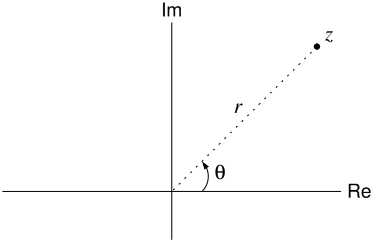
where x and y are real numbers. The quantities x and y are called the **real part** and **imaginary part** of z, and are denoted as $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively.
- The **polar form** of the complex number z expresses z in the form

$$z = r(\cos\theta + j\sin\theta) \quad \text{or equivalently} \quad z = re^{j\theta},$$

where r and θ are real numbers and $r \geq 0$. The quantities r and θ are called the **magnitude** and **argument** of z, and are denoted as $|z|$ and $\arg z$, respectively. [Note: $e^{j\theta} = \cos\theta + j\sin\theta$.]

- Since $e^{j\theta} = e^{j(\theta+2\pi k)}$ for all real θ and all integer k, the argument of a complex number is only uniquely determined to within an additive multiple of 2π.
- The **principal argument** of a complex number z, denoted $\operatorname{Arg} z$, is the particular value θ of $\arg z$ that satisfies $-\pi < \theta \leq \pi$.
- The principal argument of a complex number (excluding zero) is ***unique***.





Cartesian form:

$$z = x + jy$$

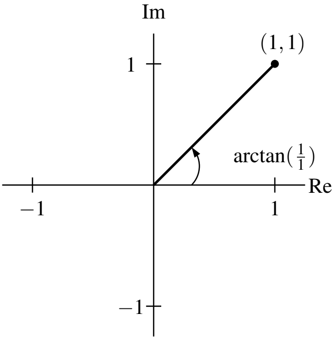
where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$

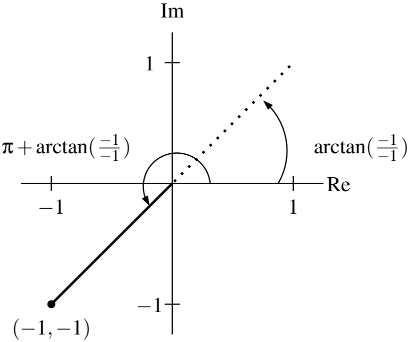
Polar form:

$$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

where $r = |z|$ and $\theta = \arg z$

- The range of the arctan function is $-\pi/2$ (exclusive) to $\pi/2$ (exclusive).
- Consequently, the arctan function always yields an angle in either the first or fourth quadrant.





The atan2 Function

- The angle θ that a vector from the origin to the point (x, y) makes with the positive x axis is given by $\theta = \operatorname{atan2}(y, x)$, where

$$\operatorname{atan2}(y,x) \triangleq \begin{cases} \arctan(y/x) & x > 0 \\ \pi/2 & x = 0 \text{ and } y > 0 \\ -\pi/2 & x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & x < 0 \text{ and } y < 0. \end{cases}$$

- The range of the atan2 function is from $-\pi$ (exclusive) to π (inclusive).
- For the complex number z expressed in Cartesian form $x + jy$, $\operatorname{Arg} z = \operatorname{atan2}(y, x)$.
- Although the atan2 function is quite useful for computing the principal argument (or argument) of a complex number, it is ***not advisable to memorize*** the definition of this function. It is better to simply understand what this function is doing (namely, intelligently applying the arctan function).

Conversion Between Cartesian and Polar Form

- Let z be a complex number with the Cartesian and polar form representations given respectively by

$$z = x + jy \quad \text{and} \quad z = re^{j\theta}.$$

- To convert from ***polar to Cartesian*** form, we use the following identities:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

- To convert from ***Cartesian to polar*** form, we use the following identities:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \operatorname{atan2}(y, x) + 2\pi k,$$

  where k is an arbitrary integer.

- Since the $\operatorname{atan2}$ function simply amounts to the intelligent application of the $\arctan$ function, instead of memorizing the definition of the $\operatorname{atan2}$ function, one should simply ***understand*** how to use the $\arctan$ function to achieve the same result.

Properties of Complex Numbers

- For complex numbers, addition and multiplication are ***commutative***. That is, for any two complex numbers z_1 and z_2,

$$z_1 + z_2 = z_2 + z_1 \quad \text{and}$$
$$z_1 z_2 = z_2 z_1.$$

- For complex numbers, addition and multiplication are ***associative***. That is, for any three complex numbers z_1, z_2, and z_3,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \text{and}$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

- For complex numbers, the ***distributive*** property holds. That is, for any three complex numbers z_1, z_2, and z_3,

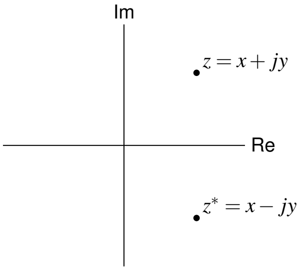
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Conjugation

- The **conjugate** of the complex number $z = x + jy$ is denoted as z^* and defined as

$$z^* = x - jy.$$

- Geometrically, the conjugation operation reflects a point in the complex plane about the real axis.
- The geometric interpretation of the conjugate is illustrated below.



Properties of Conjugation

- For every complex number z, the following identities hold:

$$\begin{aligned} \left|z^*\right| &= |z|, \\ \arg z^* &= -\arg z, \\ zz^* &= |z|^2, \\ \operatorname{Re} z &= \tfrac{1}{2}(z+z^*), \quad \text{and} \\ \operatorname{Im} z &= \tfrac{1}{2j}(z-z^*). \end{aligned}$$

- For all complex numbers z_1 and z_2, the following identities hold:

$$\begin{aligned} (z_1+z_2)^* &= z_1^* + z_2^*, \\ (z_1 z_2)^* &= z_1^* z_2^*, \quad \text{and} \\ (z_1/z_2)^* &= z_1^*/z_2^*. \end{aligned}$$

Addition

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2). \end{aligned}$$

- That is, to add complex numbers expressed in Cartesian form, we simply add their real parts and add their imaginary parts.
- ***Polar form:*** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= (r_1 \cos\theta_1 + jr_1 \sin\theta_1) + (r_2 \cos\theta_2 + jr_2 \sin\theta_2) \\ &= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2). \end{aligned}$$

- That is, to add complex numbers expressed in polar form, we first rewrite them in Cartesian form, and then add their real parts and add their imaginary parts.
- For the purposes of addition, it is easier to work with complex numbers expressed in Cartesian form.

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1 x_2 + jx_1 y_2 + jx_2 y_1 - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1). \end{aligned}$$

- That is, to multiply two complex numbers expressed in Cartesian form, we use the distributive law along with the fact that $j^2 = -1$.
- ***Polar form:*** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1 + \theta_2)}.$$

- That is, to multiply two complex numbers expressed in polar form, we use exponent rules.
- For the purposes of multiplication, it is easier to work with complex numbers expressed in polar form.

Division

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2}$$
$$= \frac{x_1 x_2 - jx_1 y_2 + jx_2 y_1 + y_1 y_2}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}.$$

- That is, to compute the quotient of two complex numbers expressed in Cartesian form, we convert the problem into one of division by a real number.
- ***Polar form:*** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}.$$

- That is, to compute the quotient of two complex numbers expressed in polar form, we use exponent rules.
- For the purposes of division, it is easier to work with complex numbers expressed in polar form.

- For any complex numbers z_1 and z_2, the following identities hold:

$$\left|z_1z_2\right| = \left|z_1\right|\left|z_2\right|,$$

$$\left|\frac{z_1}{z_2}\right| = \frac{\left|z_1\right|}{\left|z_2\right|} \quad \text{for } z_2 \neq 0,$$

$$\arg z_1z_2 = \arg z_1 + \arg z_2, \quad \text{and}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0.$$

- The above properties trivially follow from the polar representation of complex numbers.

- **Euler's relation.** For all real θ,

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

- From Euler's relation, we can deduce the following useful identities:

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{and}$$
$$\sin\theta = \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}).$$

- **De Moivre's theorem.** For all real θ and all ***integer*** n,

$$e^{jn\theta} = \left(e^{j\theta}\right)^n.$$

[Note: This relationship does not necessarily hold for ***real*** n.]

Roots of Complex Numbers

- Every complex number $z = re^{j\theta}$ (where $r = |z|$ and $\theta = \arg z$) has n distinct ***nth roots*** given by

$$\sqrt[n]{r}e^{j(\theta+2\pi k)/n} \quad \text{for } k = 0, 1, \ldots, n-1.$$

- For example, 1 has the two distinct square roots 1 and -1.

- Consider the equation

$$az^2 + bz + c = 0,$$

  where a, b, and c are real, z is complex, and $a \neq 0$.
- The roots of this equation are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- This formula is often useful in factoring quadratic polynomials.
- The quadratic $az^2 + bz + c$ can be factored as $a(z - z_0)(z - z_1)$, where

$$z_0 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

- A **complex function** maps complex numbers to complex numbers. For example, the function $F(z) = z^2 + 2z + 1$, where z is complex, is a complex function.
- A complex **polynomial function** is a mapping of the form

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where z, $a_0, a_1, \ldots, a_n$ are complex.

- A complex **rational function** is a mapping of the form

$$F(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m},$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ and z are complex.

- Observe that a polynomial function is a special case of a rational function.
- Herein, we will mostly focus our attention on polynomial and rational functions.

Continuity

- A function F is said to be **continuous at a point** z_0 if $F(z_0)$ is defined and given by

$$F(z_0) = \lim_{z \to z_0} F(z).$$

- A function that is continuous at every point in its domain is said to be **continuous**.
- Polynomial functions are continuous everywhere.
- Rational functions are continuous everywhere except at points where the denominator polynomial becomes zero.

Differentiability

- A function F is said to be **differentiable at a point** $z = z_0$ if the limit

$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0}$$

  exists. This limit is called the **derivative** of F at the point $z = z_0$.
- A function is said to be **differentiable** if it is differentiable at every point in its domain.
- The rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. If $F'(z_0)$ and $G'(z_0)$ exist, then
  1. $(aF)'(z_0) = aF'(z_0)$ for any complex constant a;
  2. $(F+G)'(z_0) = F'(z_0) + G'(z_0)$;
  3. $(FG)'(z_0) = F'(z_0)G(z_0) + F(z_0)G'(z_0)$;
  4. $(F/G)'(z_0) = \frac{G(z_0)F'(z_0) - F(z_0)G'(z_0)}{G(z_0)^2}$; and
  5. if $z_0 = G(w_0)$ and $G'(w_0)$ exists, then the derivative of $F(G(z))$ at w_0 is $F'(z_0)G'(w_0)$ (i.e., the chain rule).
- A polynomial function is differentiable everywhere.
- A rational function is differentiable everywhere except at the points where its denominator polynomial becomes zero.

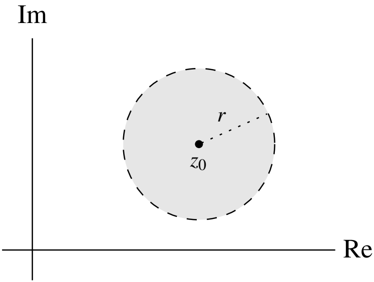
Open Disks

- An **open disk** in the complex plane with center z_0 and radius r is the set of complex numbers z satisfying

$$|z - z_0| < r,$$

where r is a strictly positive real number.

- A plot of an open disk is shown below.



Analyticity

- A function is said to be **analytic at a point** z_0 if it is differentiable at every point in an open disk about z_0.
- A function is said to be **analytic** if it is analytic at every point in its domain.
- A polynomial function is analytic everywhere.
- A rational function is analytic everywhere, except at the points where its denominator polynomial becomes zero.

- If a function F is zero at the point z_0 (i.e., $F(z_0) = 0$), F is said to have a **zero** at z_0.
- If a function F is such that $F(z_0) = 0, F^{(1)}(z_0) = 0, \ldots, F^{(n-1)}(z_0) = 0$ (where $F^{(k)}$ denotes the kth order derivative of F), F is said to have an **nth order zero** at z_0.
- A point at which a function fails to be analytic is called a **singularity**.
- Polynomials do not have singularities.
- Rational functions can have a type of singularity called a pole.
- If a function F is such that $G(z) = 1/F(z)$ has an nth order zero at z_0, F is said to have an **nth order pole** at z_0.
- A pole of first order is said to be **simple**, whereas a pole of order two or greater is said to be **repeated**. A similar terminology can also be applied to zeros (i.e., **simple zero** and **repeated zero**).

Zeros and Poles of a Rational Function

- Given a rational function F, we can always express F in factored form as

$$F(z) = \frac{K(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_M)^{\alpha_M}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_N)^{\beta_N}},$$

  where K is complex, $a_1, a_2, \ldots, a_M, b_1, b_2, \ldots, b_N$ are distinct complex numbers, and $\alpha_1, \alpha_2, \ldots, \alpha_M$ and $\beta_1, \beta_2, \ldots, \beta_N$ are strictly positive integers.
- One can show that F has ***poles*** at $b_1, b_2, \ldots, b_N$ and ***zeros*** at $a_1, a_2, \ldots, a_M$.
- Furthermore, the kth pole (i.e., b_k) is of ***order*** β_k, and the kth zero (i.e., a_k) is of ***order*** α_k.
- When plotting zeros and poles in the complex plane, the symbols “o” and “x” are used to denote zeros and poles, respectively.

Part 14

Partial Fraction Expansions (PFEs)

- Sometimes it is beneficial to be able to express a rational function as a sum of ***lower-order*** rational functions.
- This can be accomplished using a type of decomposition known as a partial fraction expansion.
- Partial fraction expansions are often useful in the calculation of inverse Laplace transforms, inverse z transforms, and inverse CT/DT Fourier transforms.

Strictly-Proper Rational Functions

- Consider a rational function

$$F(v) = \frac{\alpha_m v^m + \alpha_{m-1} v^{m-1} + \ldots + \alpha_1 v + \alpha_0}{\beta_n v^n + \beta_{n-1} v^{n-1} + \ldots + \beta_1 v + \beta_0}.$$

- The function F is said to be **strictly proper** if $m < n$ (i.e., the order of the numerator polynomial is strictly less than the order of the denominator polynomial).
- Through polynomial long division, any rational function can be written as the sum of a polynomial and a strictly-proper rational function.
- A ***strictly-proper*** rational function can be expressed as a sum of lower-order rational functions, with such an expression being called a partial fraction expansion.

Section 14.1

PFEs for First Form of Rational Functions

- Any rational function can be expressed in the form of

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_0}{v^n + b_{n-1} v^{n-1} + \ldots + b_0}.$$

- Furthermore, the denominator polynomial $D(v) = v^n + b_{n-1} v^{n-1} + \ldots + b_0$ in the above expression for $F(v)$ can be factored to obtain

$$D(v) = (v - p_1)^{q_1} (v - p_2)^{q_2} \cdots (v - p_n)^{q_n},$$

  where the p_k are distinct and the q_k are integers.
- If F has only simple poles, $q_1 = q_2 = \cdots = q_n = 1$.
- Suppose that F is strictly proper (i.e., $m < n$).
- In the determination of a partial fraction expansion of F, there are ***two cases*** to consider:
  1. F has ***only simple poles***; and
  2. F has ***at least one repeated pole***.

- Suppose that the (rational) function F has only simple poles.
- Then, the denominator polynomial D for F is of the form

$$D(v) = (v - p_1)(v - p_2) \cdots (v - p_n),$$

where the p_k are distinct.

- In this case, F has a partial fraction expansion of the form

$$F(v) = \frac{A_1}{v - p_1} + \frac{A_2}{v - p_2} + \ldots + \frac{A_{n-1}}{v - p_{n-1}} + \frac{A_n}{v - p_n},$$

where

$$A_k = (v - p_k)F(v)\big|_{v=p_k}.$$

- Note that the (simple) pole p_k contributes a single term to the partial fraction expansion.

- Suppose that the (rational) function F has at least one repeated pole.
- One can show that, in this case, F has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{1,1}}{v-p_1} + \frac{A_{1,2}}{(v-p_1)^2} + \ldots + \frac{A_{1,q_1}}{(v-p_1)^{q_1}}\right] + \left[\frac{A_{2,1}}{v-p_2} + \ldots + \frac{A_{2,q_2}}{(v-p_2)^{q_2}}\right] + \ldots + \left[\frac{A_{P,1}}{v-p_P} + \ldots + \frac{A_{P,q_P}}{(v-p_P)^{q_P}}\right],$$

where

$$A_{k,\ell} = \frac{1}{(q_k-\ell)!}\left[\left[\tfrac{d}{dv}\right]^{q_k-\ell}\left[(v-p_k)^{q_k}F(v)\right]\right]\Big|_{v=p_k}.$$

- Note that the q_kth-order pole p_k contributes q_k terms to the partial fraction expansion.
- Note that $n! = (n)(n-1)(n-2)\cdots(1)$ and $0! = 1$.

Section 14.2

PFEs for Second Form of Rational Functions

- Any rational function can be expressed in the form of

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_1 v + a_0}{b_n v^n + b_{n-1} v^{n-1} + \ldots + b_1 v + 1}.$$

- Furthermore, the denominator polynomial $D(v) = b_n v^n + b_{n-1} v^{n-1} + \ldots + b_1 v + 1$ in the above expression for $F(v)$ can be factored to obtain

$$D(v) = (1 - p_1^{-1} v)^{q_1} (1 - p_2^{-1} v)^{q_2} \cdots (1 - p_n^{-1} v)^{q_n},$$

  where the p_k are distinct and the q_k are integers.
- If F has only simple poles, $q_1 = q_2 = \cdots = q_n = 1$.
- Suppose that F is strictly proper (i.e., $m < n$).
- In the determination of a partial fraction expansion of F, there are ***two cases*** to consider:
  1. F has ***only simple poles***; and
  2. F has ***at least one repeated pole***.

- Suppose that the (rational) function F has only simple poles.
- Then, the denominator polynomial D for F is of the form

$$D(v) = (1 - p_1^{-1}v)(1 - p_2^{-1}v)\cdots(1 - p_n^{-1}v),$$

where the p_k are distinct.
- In this case, F has a partial fraction expansion of the form

$$F(v) = \frac{A_1}{1 - p_1^{-1}v} + \frac{A_2}{1 - p_2^{-1}v} + \ldots + \frac{A_{n-1}}{1 - p_{n-1}^{-1}v} + \frac{A_n}{1 - p_n^{-1}v},$$

where

$$A_k = (1 - p_k^{-1}v)F(v)\big|_{v=p_k}.$$

- Note that the (simple) pole p_k contributes a single term to the partial fraction expansion.

- Suppose that the (rational) function F has at least one repeated pole.
- One can show that, in this case, F has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{1,1}}{1-p_1^{-1}v} + \frac{A_{1,2}}{(1-p_1^{-1}v)^2} + \ldots + \frac{A_{1,q_1}}{(1-p_1^{-1}v)^{q_1}}\right] + \left[\frac{A_{2,1}}{1-p_2^{-1}v} + \ldots + \frac{A_{2,q_2}}{(1-p_2^{-1}v)^{q_2}}\right] + \ldots + \left[\frac{A_{P,1}}{1-p_P^{-1}v} + \ldots + \frac{A_{P,q_P}}{(1-p_P^{-1}v)^{q_P}}\right],$$

where

$$A_{k,\ell} = \frac{1}{(q_k-\ell)!}(-p_k)^{q_k-\ell}\left[\left[\tfrac{d}{dv}\right]^{q_k-\ell}\left[(1-p_k^{-1}v)^{q_k}F(v)\right]\right]\Big|_{v=p_k}.$$

- Note that the q_kth-order pole p_k contributes q_k terms to the partial fraction expansion.
- Note that $n! = (n)(n-1)(n-2)\cdots(1)$ and $0! = 1$.

Part 15

Miscellany

Sum of Arithmetic and Geometric Sequences

- The sum of the arithmetic sequence $a, a+d, a+2d, \ldots, a+(n-1)d$ is given by

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n[2a+d(n-1)]}{2}.$$

- The sum of the geometric sequence $a, ra, r^2a, \ldots, r^{n-1}a$ is given by

$$\sum_{k=0}^{n-1} r^k a = a\frac{r^n-1}{r-1} \quad \text{for } r \neq 1.$$

- The sum of the infinite geometric sequence $a, ra, r^2a, \ldots$ is given by

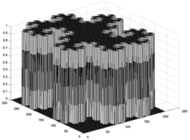
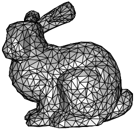
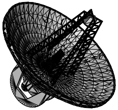
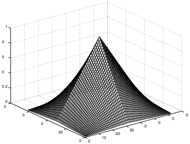
$$\sum_{k=0}^{\infty} r^k a = \frac{a}{1-r} \quad \text{for } |r| < 1.$$

Part 16

Epilogue

Other Courses Offered by the Author of These Lecture Slides

- If you did not suffer permanent emotional scarring as a result of using these lecture slides and you happen to be a student at the University of Victoria, you might wish to consider taking another one of the courses developed by the author of these lecture slides:
  - *ECE 486: Multiresolution Signal and Geometry Processing with C++*
  - *SENG 475: Advanced Programming Techniques for Robust Efficient Computing*
- For further information about the above courses (including the URLs for web sites of these courses), please refer to the slides that follow.

ECE 486/586: Multiresolution Signal and Geometry Processing with C++

- normally offered in Summer (May-August) term; only prerequisite ECE 310
- subdivision surfaces and subdivision wavelets
  - 3D computer graphics, animation, gaming (Toy Story, Blender software)
  - geometric modelling, visualization, computer-aided design
- multirate signal processing and wavelet systems
  - sampling rate conversion (audio processing, video transcoding)
  - signal compression (JPEG 2000, FBI fingerprint compression)
  - communication systems (transmultiplexers for CDMA, FDMA, TDMA)
- C++ (classes, templates, standard library), OpenGL, GLUT, CGAL
- software applications (using C++)
- for more information, visit course web page:
  `http://www.ece.uvic.ca/~mdadams/courses/wavelets`

SENG 475: Advanced Programming Techniques for Robust Efficient Computing (With C++)

- advanced programming techniques for robust efficient computing explored in context of C++ programming language
- topics covered may include:
  - concurrency, multithreading, transactional memory, parallelism, vectorization; cache-efficient coding; compile-time versus run-time computation; compile-time versus run-time polymorphism; generic programming techniques; resource/memory management; copy and move semantics; exception-safe coding
- applications areas considered may include:
  - geometry processing, computer graphics, signal processing, and numerical analysis
- open to any student with necessary prerequisites, which are:
  - SENG 265 or CENG 255 or CSC 230 or CSC 349A or ECE 255 or permission of Department
- for more information, see course web site:

  `http://www.ece.uvic.ca/~mdadams/courses/cpp`