Part I

The Science of Audio The Mathematics of Audio

1

Discrete Fourier Transform

We have considered the frequency response (amplitude and phase) of an LTI system with sampling rate f_s

$$H(f) = H(z)|_{z=exp(j2\pi f/f_s)}, -f_s/2 < f < f_s/2 \tag{1.1}$$

To calculate and plot $H(f)$ using a computer, we need to calculate $H(f)$ at discrete frequencies in the range $-f_s/2 < f < f_s/2$. We can choose to use N_0 points to cover a span of f_s, spaced at intervals of $f_s/N_0 = f_0$.

1.1 DEFINITION OF DFT

We define the Discrete Fourier Transform (DFT) as the frequency response of an LTI system sampled at N_0 samples around the whole unit circle $-f_s/2 < f < f_s/2$ with frequency resolution (step size) $f_0 = f_s/N_0$

$$\begin{aligned}
H[r] &= H(f)|_{f=rf_0=rf_s/N_0}, 0 \leq r \leq N_0 - 1 && (1.2)\\
&= H(z)|_{z=exp(j2\pi f/f_s)}|_{f=rf_s/N_0} && (1.3)\\
&= H(z)|_{z=exp(j2\pi r/N_0)} && (1.4)\\
&= \sum_{k=0}^{N_0-1} h[k]z^{-k}|_{z=exp(j2\pi r/N_0)} && (1.5)\\
&= \sum_{k=0}^{N_0-1} h[k]exp(-j2\pi rk/N_0) && (1.6)
\end{aligned}$$

Thus the DFT input is N_0 time samples $h[k]$ spaced at time intervals $T_s = 1/f_s$ which may be written as a vector $\mathbf{h}$. The DFT output is N_0 frequency samples $H[r]$. spaced at frequency intervals f_0 which may be written as a vector H.

Similarly, the inverse DFT (IDFT) is

$$h[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} H[r] exp(+j2\pi rk/N_0) \tag{1.7}$$

so the IDFT input is N_0 frequency samples $H[r]$ and the IDFT output is N_0 time samples $h[k]$.

1.2 CALCULATION OF DFT

To evaluate the expression for the DFT, we take a block of N_0 time samples $h[k]$, start with frequency $f = rf_0 = 0$ with index $r = 0$, and compute a sum over the time index k to obtain $H[r]$ for $r = 0$, and then repeat for each index r for $0 \leq r \leq N_0 - 1$. We can write the DFT

$$H[r] \quad = \quad \sum_{k=0}^{N_0-1} h[k](W_{N_0})^{rk} \tag{1.8}$$

$$\tag{1.9}$$

where we use the shorthand notation

$$W_{N_0} = exp(-j2\pi/N_0) \tag{1.10}$$

For example, for $N_0 = 4$, we can write

$$H[r] = h[0](W_{N_0})^{0 \cdot r} + h[1](W_{N_0})^{1 \cdot r} + h[2](W_{N_0})^{2 \cdot r} + h[3](W_{N_0})^{3 \cdot r} \tag{1.11}$$

and evaluate $H[r]$ for each r in the range $0 \leq r \leq N_0 - 1$.

An equivalent way to compute the DFT is write it as a matrix multiplication of a vector $\mathbf{h}$ with N_0 elements with a $N_0 \times N_0$ matrix $\mathbf{W_{DFT}}$ to obtain an N_0 element vector H. Thus $H = \mathbf{Wh}$, where the matrix for $N_0 = 4$ is written

$$\mathbf{W_{DFT}} \quad = \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & (W_4)^2 & (W_4)^3 \\ 1 & (W_4)^2 & (W_4)^4 & (W_4)^6 \\ 1 & (W_4)^3 & (W_4)^6 & (W_4)^9 \end{bmatrix} \tag{1.12}$$

The IDFT $\mathbf{h} = \mathbf{W_{IDFT}H}$, where the matrix

$$\mathbf{W} \quad = \quad \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & (W_4)^{-2} & (W_4)^{-3} \\ 1 & (W_4)^{-2} & (W_4)^{-4} & (W_4)^{-6} \\ 1 & (W_4)^{-3} & (W_4)^{-6} & (W_4)^{-9} \end{bmatrix} \tag{1.13}$$

1.3 INTERPRETATION OF DFT

The DFT may be interpreted is as a sampled version of the continuous time Fourier transform by writing

$$H(f) = \int h(t) exp(-j2\pi ft) dt \tag{1.14}$$

substituting $f = rf_0$ for N_0 frequency samples spaced by frequency resolution f_0 and substituting $t = kT_s$ for N_0 time samples spaced by time resolution (sampling time) T_s We then write use $f_0 = f_s/N_0$ and $f_s = 1/T_s$ to obtain

$$\begin{aligned} H[r] = H(f = rf_0) &= \sum_{k=0}^{N_0-1} h(t = kT_s) exp(-j2\pi r f_0 k T_s) & (1.15) \\ &= \sum_{k=0}^{N_0-1} h(t = kT_s) exp(-j2\pi r f_s k T_s / N_0) & (1.16) \\ &= \sum_{k=0}^{N_0-1} h(k) exp(-j2\pi rk/N_0) & (1.17) \end{aligned}$$

using $f_0 = f_s/N_0$ and $f_s = 1/T_s$.

Thus the DFT is a numerical way to compute a Fourier transform using N_0 samples of the time domain impulse response over the time span $N_0 T_s = T_0$ to obtain N_0 samples of the frequency response over the frequency span $N_0 f_0 = f_s$

Similarly, the inverse DFT is a sampled version of the inverse Fourier transform

$$h(t) = \int H(f) exp(+j2\pi ft) df \tag{1.18}$$

substituting $f = rf_0, t = kT_s$ to obtain

$$h[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} H[r] exp(+j2\pi rk/N_0) \tag{1.19}$$

The DFT implicitly assumes that the sampled time signal is periodic with period $T_0 = N_0 T_s$. The N_0 time samples spaced by sampling time (time resolution) T_s are repeated at intervals T_0. We already know that the sampled frequency response is periodic with period $f_s = N_0 f_0$ (since we go around the unit circle). The N_0 frequency samples spaced by frequency resolution f_0 are repeated at intervals f_s.

1.3.1 Zero-padding

If we require a frequency resolution f_0 for a signal sampled at f_s, we need $N_0 = f_s/f_0$ time samples. Thus the sampled time signal will have a duration of $N_0 T_s = N_0/f_s = 1/f_0 = T_0$. However, if the sampled time signal is of short duration less than T_0, i.e. has $N < N_0$ samples, then the remaining $N_0 - N$ samples are simply set to zero.

2

Complex variables and inverse z-transforms

2.1 COMPLEX CONTOUR INTEGRATION

Where as the z-transform is given by a straightforward sum

$$H(z) = \sum_{k=0}^{\infty} h[k]z^{-k} \tag{2.1}$$

the inverse$z$-transform is given by an complex contour integral of form

$$h[k] = \frac{1}{2\pi j}\oint_c H(z)z^{k-1}dz \tag{2.2}$$

The integral has a complex variable z and the closed path c is a circle which encloses all poles of $H(z)z^{k-1}$ We will show what this means, and how to evaluate such integrals.

For real integrals of the form

$$I = \int_{x_1}^{x_2} f(x)dx \tag{2.3}$$

the path from a to b is along the real axis. For complex integrals of the form

$$I = \int_{z_1}^{z_2} g(z)dz \tag{2.4}$$

the limits of integration $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$ are points in the complex plane. The path between these points is not unique, and can be a straight line or a curve. If $z_1 = z_2$, the path may be of zero length, or the path may be a circle or other closed path which begins and ends at z_1. Thus if $z_1 = z_2$, the complex integral is not necessarily zero, as it would be for a real integral with $x_1 = x_2$. For the inverse z-transform,

$$h[k] = \frac{1}{2\pi j}\oint_c g(z)dz \tag{2.5}$$

with $g(z) = H(z)z^{k-1}$, we have a closed path, but the values of $h[k]$ are not necessarily zero.

2.1.1 Integration along a path

To evaluate the complex integral, we follow 3 steps.

First we define the path in the complex plane between the limits of integration z_1, z_2 in terms of a parameter t

$$z(t) = x(t) + jy(t), 0 \leq t \leq 1 \tag{2.6}$$

where t is real, so that $z(t=0) = z_1 = a_1 + jb_1$ and $z(t=1) = z_2 = a_2 + jb_2$.

For example, if the path is a straight line between $a_1 + jb_1$ and $a_2 + jb_2$, then we can write

$$z(t) = (a_2 - a_1)t + a_1 + j(b_2 - b_1)t + jb_1 \tag{2.7}$$

so that

$$x(t) = (a_2 - a_1)t + a_1, y(t) = j(b_2 - b_1)t + jb_2 \tag{2.8}$$

If the path is a closed circle of radius a and center b defined by $|z - b| = a$, then we can write

$$z(t) = b + ae^{jt}, 0 \leq t \leq 2\pi \tag{2.9}$$

Then we write the complex function $g(z)$ as real functions of real variables x and y. We write

$$z = x + jy, g(z) = u(x,y) + jv(x,y) \tag{2.10}$$

where u and v are real functions of the two real variables x, y. For example, if

$$g(z) = z^2 = (x+jy)^2 = x^2 - y^2 + j2xy = u(x,y) + jv(x,y) \tag{2.11}$$

then

$$u(x,y) = x^2 - y^2, v(x,y) = 2xy\mathrm{NOT}j2xy \tag{2.12}$$

We then write the integral

$$\int_c g(z)dz \quad = \quad \int_{t=0}^{1} g[z(t)]\frac{dz}{dt}dt \tag{2.13}$$

$$\begin{aligned} &= \int_{t=0}^{1} (u+jv)\left(\frac{dx}{dt}+j\frac{dy}{dt}\right)dt && (2.14)\\ &= \int_{t=0}^{1} u\frac{dx}{dt}dt + \int_{t=0}^{1} v\frac{dy}{dt}dt && (2.15)\\ &+ j\left[\int_{t=0}^{1} u\frac{dy}{dt}dt + \int_{t=0}^{1} v\frac{dx}{dt}dt\right] && (2.16) \end{aligned}$$

For example,consider

$$I = \int_{z=-1}^{j} z^2 dz \tag{2.17}$$

along a straight line between the integration limits.

Step 1, define path. Lower limit $a_1 + jb_1 = -1$ so $a_1 = -1, b_1 = 0$ Upper limit $a_2 + jb_2 = j$ so $a_2 = 0, b_2 = 1$ Path

$$z(t) = (a_2 - a_1)t + a_1 + j(b_2 - b_1)t + jb_1 = t + -1 + jt + 0 = -1 + t + jt \tag{2.18}$$

$$x(t) = -1 + t, \frac{dx}{dt} = 1, y(t) = t, \frac{dy}{dt} = 1 \tag{2.19}$$

Step 2, from above for $g(z) = z^2$,

$$u(x, y) = x^2 - y^2 = (-1 + t)^2 - t^2 = 1 - 2t \tag{2.20}$$

$$v(x, y) = 2xy = 2(-1 + t)t = -2t + t^2 \tag{2.21}$$

Step 3, substitute in general formula with u, v

$$\begin{aligned} I &= \int_{t=0}^{1} (u+jv)\left(\frac{dx}{dt}+j\frac{dy}{dt}\right)dt && (2.22)\\ &= \int_{t=0}^{1} [1-2t+j(2t+t^2)][1+j]dt && (2.23) \end{aligned}$$

Shorter solution not using u, v. Step 1, path

$$z(t) = -1 + t + jt, \frac{dz}{dt} = 1 + j \tag{2.24}$$

Skip step 2. Step 3, general formula with z, t

$$\begin{aligned} \int_c g(z)dz &= \int_{t=0}^{1} g[z(t)]\frac{dz}{dt}dt && (2.25)\\ &= \int_{t=0}^{1} [-1+t+jt]^2(1+j)dt && (2.26)\\ & && (2.27) \end{aligned}$$

The complex contour integrals

$$I = \int_c g(z)dz \tag{2.28}$$

are independent of the path, provided that the function $g(z)$ is analytic in a simply connected domain [1] D, i.e. defined and differentiable at all points in D. Usually the domain D is the entire complex plane, except for the point at infinity. If $g(z)$ is analytic, then

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y} \tag{2.29}$$

For analytic functions in simply connected somain D with a path in D, we can use methods of real calculus to evaluate integrals, and write

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) \tag{2.30}$$

where $dF(z)/dz = f(z)$. Integrals around a closed path where the lower limit equals the upper limit will be zero. This is called Cauchy's theorem.

For a second example, consider

$$I=\int_c \frac{1}{z}dz \tag{2.31}$$

where the path c is the unit circle counterclockwise. Note that for this example, $g(z)$ has a pole at $z=0$ and thus is not analytic in a simply connected domain D and thus the integral around a closed path is not necessarily zero.

Step 1, define path. Unit circle

$$z(t)=e^{jt}, 0\le t\le 2\pi, \frac{dz}{dt}=je^{jt} \tag{2.32}$$

Lower limit 0, upper limit 2π. Skip step 2. Step 3

$$\begin{aligned}
\int_c g(z)dz &= \int_{t=0}^{2\pi} g[z(t)]\frac{dz}{dt}dt && (2.33)\\
&= \int_{t=0}^{2\pi} \frac{1}{e^{jt}} je^{jt}dt && (2.34)\\
&= \int_{t=0}^{2\pi} jdt = 2\pi j && (2.35)\\
& && (2.36)
\end{aligned}$$

Using the same approach

$$I=\int_c z^m dz = \delta(m+1) \tag{2.37}$$

[1] A simply connected domain is a region in the complex plane where every simple closed curve (without self-intersections) encloses only points in D. A donut shaped region with a hole in it is NOT simply-connected.

i.e. is zero for all values of m except $m = -1$.

For a third example, consider the inverse z-transform integral where $g(z) = H(z)z^{k-1}$ a ratio of polynomials

$$h[k] = \frac{1}{2\pi j} \oint_c g(z)dz \tag{2.38}$$

where the path c encloses all poles of $g(z)$. In this case, we can use partial fractions to reduce $g(z)$ to terms of the form

$$\frac{Cz}{(z-p)^m} \tag{2.39}$$

We will get a different set of partial fractions terms for each different value of k. We then do a change of variable $u = z - p$ and change the path c to a circle centered at p which encloses all poles of $g(z)$. This is ok, because the integral depends only on the poles inside the circle and thus c can be any curve (circle, square or other shape) which encloses the poles. Then we write for each partial fraction term

$$\oint_c \frac{u+p}{u^m} du \tag{2.40}$$

and substitute the path to be a circle centered at p

$$u(t) = e^{jt}, 0 \leq t \leq 2\pi, \frac{du}{dt} = je^{jt} \tag{2.41}$$

and use the same approach is in the second example above.

2.1.2 Integration by method of residues

For functions which have poles (and thus are not analytic) we can avoid the method of integration along a path, by using the so-called Residue theorem

$$\int_c g(z)dz = 2\pi j \sum_{res} g(z) \tag{2.42}$$

the sum of all residues at poles inside c.

The residues are calculated using the following result. First order pole

$$res_{z=a} g(z) = lim_{z \to a}[(z-a)g(z)] \tag{2.43}$$

if $g(z) = \frac{P(z)}{Q(z)}$, then

$$res_{z=a} g(z) = \frac{P(a)}{Q'(a)} \tag{2.44}$$

Second order pole

$$res_{z=a} g(z) = lim_{z \to a} \frac{d}{dz}[(z-a)^2 g(z)] \tag{2.45}$$

Third order pole

$$res_{z=a}g(z) = lim_{z\to a}\frac{1}{2}\frac{d^2}{dz^2}[(z-a)^3 g(z)] \tag{2.46}$$

mth order pole

$$res_{z=a}g(z) = lim_{z\to a}\frac{1}{m!}\frac{d^{(m-1)}}{dz^{(m-1)}}[(z-a)^m g(z)] \tag{2.47}$$

The Residue theorem is very useful for evaluating inverse z-transforms

$$h[k] = \frac{1}{2\pi j}\int_c H(z)z^{k-1}dz \tag{2.48}$$

by defining $g(z) = H(z)z^{k-1}$. where $H(z)$ is the transfer function.

Note that to find $h[k]$, we have to evaluate a different integral for each value of k. Also note that for $k = 0$, there is an extra pole at $z = 0$ in addition to the poles in $H(z)$.

2.2 LAURENT SERIES

Since the z-transform is obtained directly from a series expansion in powers of z^{-1},

$$H(z) = \sum_{k=0}^{\infty} h[k]z^{-k} \tag{2.49}$$

the inverse z-transform can be obtained by expanding $H(z)$ in powers of z^{-1} (Expansion of $H(z)$ in powers of z is called a Taylor series, expansion in powers of z^{-1} is called a Laurent series).

To do this series expansion, we break $H(z)/z$ into partial fractions to get terms of the form

$$\frac{H(z)}{z} = \frac{C_1}{z-p_1} + \frac{C_2}{z-p_2} + ... \tag{2.50}$$

so that by multiplying both sides by z, $H(z)$ is the sum of terms of the form

$$\frac{Cz}{z-p} = \frac{C}{1-pz^{-1}} = \frac{C}{1-w} \tag{2.51}$$

where $w = pz^{-1}$.

Then, for each partial fraction term, we use the geometric series

$$\frac{C}{1-w} = C\sum_{k=0}^{\infty} w^k, |w| < 1 \tag{2.52}$$

$$\frac{C}{(1-w)^2} = C\sum_{k=0}^{\infty}(k+1)w^k = C\sum_{n=1}^{\infty} nw^{(n-1)}, |w| < 1 \tag{2.53}$$

$$\frac{C}{(1-w)^3} = C\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}w^k = C\sum_{n=2}^{\infty}\frac{n(n-1)}{2}w^{(n-2)}, |w|<1 \quad (2.54)$$

When we add up all the terms from each partial fraction term, we obtain the Laurent series for $H(z)$ in powers of z^{-1}. We can then write

The series for each partial fraction term converges for $|w| < 1$ or $|pz^{-1}| < 1$ or $|p| < |z|$ or $|z| > |p|$. The circle $|z| = p$ is the boundary of the region of convergence, and we define the radius of the circle p to be the radius of convergence (ROC).

We can get $H(z)$ in powers of z by breaking $H(z)$ into partial fractions to get a sum of terms of the form

$$\frac{C}{z-p} == \frac{-C/p}{1-z/p} = \frac{-C/p}{1-w} \quad (2.55)$$

where $w = z/p$. We then use the geometric series to obtain the Taylor series for $H(z)$ in powers of z. The Taylor series converges for $|w| < 1$ or $|z/p| < 1$ or $|z| < |p|$.

Thus we have two different series for each partial fraction term of $H(z)$, the Laurent series in powers of z^{-1} that converges for $|z| > |p|$, and the Taylor series in powers of z that converges for $|z| < |p|$.

For the purpose of finding inverse z-transforms, we choose the region $|z| > p_{max}$, where p_{max} is the pole of $H(z)$ with the largest magnitude. We add up all the Laurent series terms from each partial fraction term, and obtain the overall Laurent series for $H(z)$ in powers of z^{-1}, which we can write

$$H(z) = \sum_{k=0}^{\infty} h[k]z^{-k} \quad (2.56)$$

where the terms in the impulse response $h[k]$ are the coefficients of the series.

We can find the radius of convergence p_{max} directly from the Laurent series by considering the convergence properties of the series

$$\sum_{k=0}^{\infty} h[k] \quad (2.57)$$

If

$$lim_{M\to\infty}\sum_{k=0}^{M} h[k] = h \quad (2.58)$$

then the series converges. We can use standard tests for the convergence of series. Using the ratio test, if

$$lim_{k\to\infty}|h[k+1]/h[k]| = q \quad (2.59)$$

then the ROC is $1/q$. Using the ratio test, if

$$lim_{k\to\infty}(h[k])^{1/k} = q \quad (2.60)$$

then the ROC is $1/q$.