Solution to Assignment 3

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I. Question 1

For the three filters of assignment 1, find the transfer function of $H(z)$. Write $H(z)$ as a ratio of polynomials and identify the poles and zeros. Choose a $a = 0.9$.

Solution.

(a) For the impulse response

$$h[k] = 0.5\delta[k] + \delta[k-1] + 0.5\delta[k-2]$$

its *z-transform* can be expressed as:

$$\begin{aligned} H(z) &= \sum_{k=-\infty}^{\infty} h[k] z^{-k} \\ &= 0.5\delta[0] z^0 + \delta[1-1] z^{-1} + 0.5\delta[2-2] z^{-2} \\ &= 0.5 + z^{-1} + 0.5z^{-2} \\ &= \frac{0.5(z+1)^2}{z^2} \end{aligned}$$

The above equation is equivalent to

$$H(z) = b_0 \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$

where $z_1 = z_2 = -1$ and $p_1 = p_2 = 0$ and $b_0 = 0.5$.

(b) For the impulse response

$$h[k] = \delta[k] - \delta[k-1]$$

its *z-transform* can be expressed as:

$$\begin{aligned} H(z) &= \sum_{k=-\infty}^{\infty} h[k] z^{-k} \\ &= \delta[0] z^0 - \delta[1-1] z^{-1} \\ &= 1 - z^{-1} \\ &= \frac{z-1}{z} \end{aligned}$$

The above equation is equivalent to

$$H(z) = b_0 \frac{z-z_1}{z-p_1}$$

where $z_1 = 1$ and $p_1 = 0$ and $b_0 = 1$.

(c) For the output sequence

$$y\left[n\right] \;=\; ay\left[n-1\right] + x\left[n-1\right]$$

its *z-transform* can be expressed as:

$$\begin{aligned} Y\left(z\right) &= \sum_{k=-\infty}^{\infty} y[k]z^{-k} \\ &= \sum_{k=-\infty}^{\infty} ay[k-1]z^{-k} + x[k-1]z^{-k} \end{aligned}$$

Using the linearity and time-shifting properties of the *z-transform* , we can write

$$Y\left(z\right) \;=\; aY(z)z^{-1} + X(z)z^{-1}$$

The above equation is equivalent to $(1 - az^{-1})Y(z) = X(z)z^{-1}$ Thus

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{z^{-1}}{1 - az^{-1}} \\ &= \frac{1}{z - a} \\ &= \frac{1}{z - p_1} \end{aligned}$$

where $p_1 = a = 0.9$.

II. Question 2

For the 3 filters of assignment 1, find the impulse response $h(n)$ in two ways:

1) Work out algebraically using the inverse *z-transform*, and evaluate the first few terms numerically.

2) Use the POI software to place the poles and zeros in the complex plane, and let the program calculate the impulse response. In POI, drag zeros on the z-plane from the top left, and drag poles from the top right. More details are in the POI on-line documentation under "ZPlane Design". Under the tap "analysis", choose "analysis graph", and then in the new window under the tab "graph" choose "impulse response". Under the tab

"file" choose "save values" which will save the impulse response values in a file. For more documentation of POI, see the POI on-line documentation

Compare the results of algebra with POI. The sampling rate fs is not defined in Assignment 1, the POI software lets you choose the sampling rate. Choose fs = 44,100 Hz used for music CDs.

Solution.

1) Algebraic solution

(a)

$$\begin{aligned} H(z) &= \frac{1}{2}\frac{(z+1)^2}{z^2} \\ &= 0.5 + z^{-1} + 0.5z^{-2} \end{aligned}$$

Using the table of *z-transform*, ($\delta[n] \Longrightarrow 1, \delta[n-k] \Longrightarrow z^{-k}$)

$$h[k] = 0.5\delta[k] + \delta[k-1] + 0.5\delta[k-2]$$

Evaluate $h[k]$ for different values of k

$$\begin{aligned} k = 0 : h[0] &= 0.5 \\ k = 1 : h[1] &= 1.0 \\ k = 2 : h[2] &= 0.5 \\ k = 3 : h[3] &= 0.0 \\ k = 4 : h[4] &= 0.0 \end{aligned}$$

(b)

$$\begin{aligned} H(z) &= \frac{z-1}{z} \\ &= 1 - z^{-1} \end{aligned}$$

From the table of *z-transform*,

$$h\left[k\right] = \delta\left[k\right] - \delta\left[k-1\right]$$

Evaluate $h[k]$ for different values of k

$$\begin{aligned} k=0: h[0] &= 1.0 \\ k=1: h[1] &= -1.0 \\ k=2: h[2] &= 0.0 \\ k=3: h[3] &= 0.0 \end{aligned}$$

(c)

$$H\left(z\right) = \frac{1}{z-a}$$

Using the table of *z-transform*, ($p^{k-1}u[k-1] \Longrightarrow \frac{1}{z-p}$)

$$\begin{aligned} h\left[k\right] &= a^{k-1}u[k-1] \\ &= 0.9^{k-1}u[k-1] \end{aligned}$$

Evaluate $h[k]$ for different values of k

$$\begin{aligned} k=0: h[0] &= 0.0 \\ k=1: h[1] &= 1.0 \\ k=2: h[2] &= 0.9 \\ k=3: h[3] &= 0.9^2 \\ k=4: h[3] &= 0.9^3 \end{aligned}$$

2) Results from POI software

Impulse responses of the three filters are shown in Fig.1.

We can observe that the algebraic results are the same as POI's results.

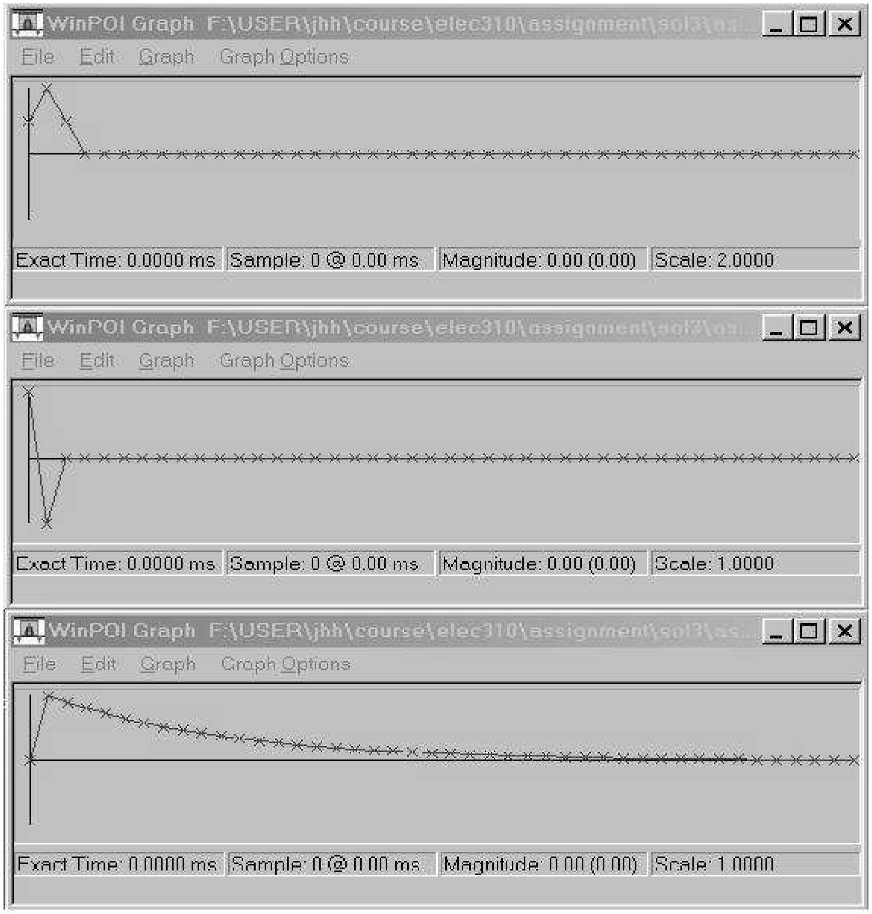

Fig. 1. Impulse responses of the three filters

III. Question 3

POI also calculates the frequency response H(z) with z along the unit circle $z = e^{j\hat{\omega}} = e^{j2\pi f/f_s}$ for $|f|$ less than $f_s/2$. The sampling rate f_s is not defined in Assignment 1, the POI software lets you choose the sampling rate. For $f_s = 44,100Hz$, calculate the frequency response for each of the 3 filters of Assignment 1. Do this two ways:

1) Use the POI software to place the poles and zeros in the complex plane, and let the program calculate the frequency response.

2) Work out the algebraic expression for the frequency response and plot using Matlab. Compare the results from POI and Matlab.

Solution.

1) Results from POI software

Frequency responses calculated from POI software are shown in Fig.2.

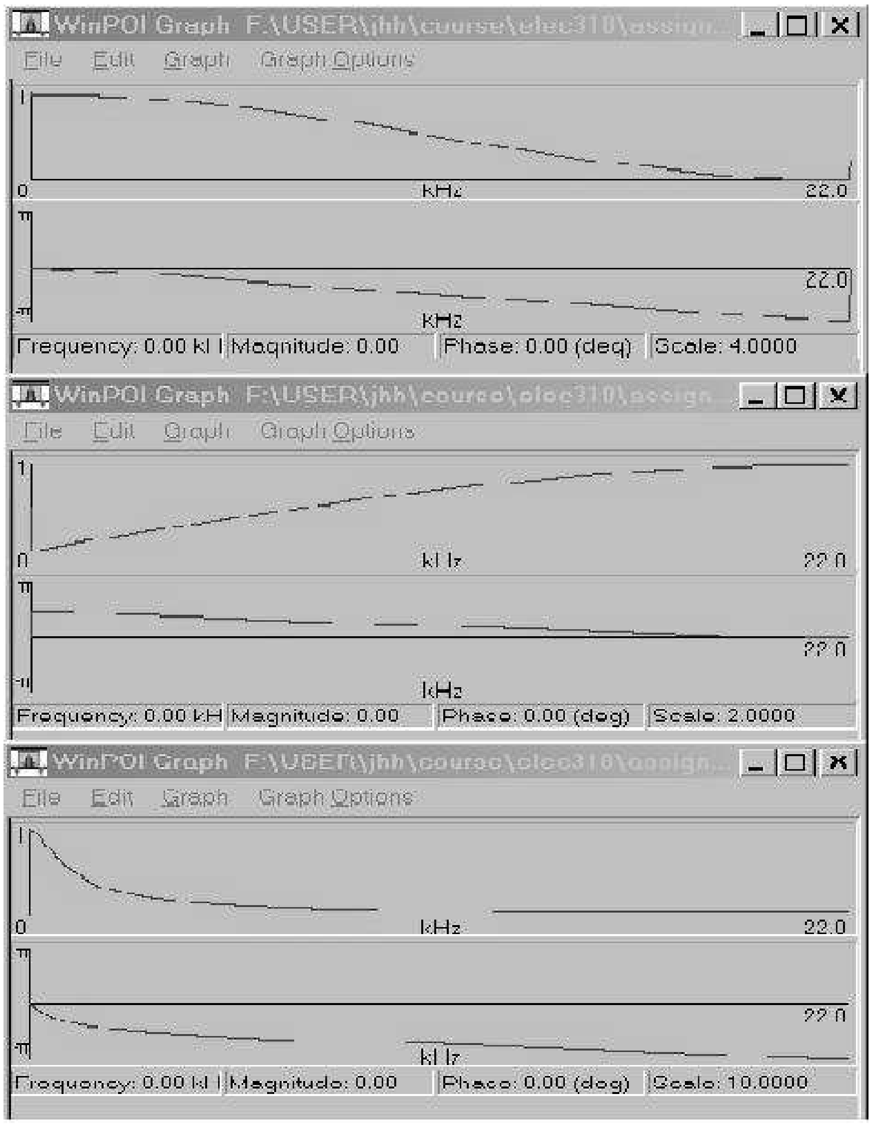

Fig. 2. Frequency responses of the three filters obtained by POI

2) Matlab plot for the algebraic expression

(a) The frequency response of an LTI system is given by the transfer function evaluated on the unit circle

$$H(f) = H(z)|_{z=e^{j\hat{\omega}}} = H(z)|_{z=e^{j2\pi f/f_s}}$$

where $\hat{\omega} = 2\pi f/f_s$ is the normalized radian frequency, and f_s is the sampling frequency.

For the first filter,

$$
\begin{aligned}
H(f) &= H(z)_{z=e^{j\hat{\omega}}} \\
&= H(z)|_{z=e^{j2\pi f/f_s}} \\
&= \frac{1}{2}\frac{(e^{j2\pi f/f_s}+1)^2}{(e^{j2\pi f/f_s})^2} \\
&= \frac{1}{2}\frac{(cos2\pi f/f_s + jsin2\pi f/f_s + 1)^2}{(e^{j2\pi f/f_s})^2} \\
&= \frac{1}{2}\frac{r_1 e^{j\phi_1}}{d_1 e^{j\theta_1}} \\
&= \frac{1}{2}\frac{r_1}{d_1}e^{j\phi_1-\theta_1}
\end{aligned}
$$

The magnitude response is

$$
|H(f)| = \frac{1}{2}\frac{r_1}{d_1}
$$

and phase response is

$$
arg(H(f)) = \phi_1 - \theta_1
$$

where

$$
\begin{aligned}
r_1 &= (cos(2\pi f/f_s)+1)^2 + sin^2(2\pi f/f_s) \\
d_1 &= 1 \\
\phi_1 &= \arctan\left(\frac{sin(2\pi f/f_s)}{cos(2\pi f/f_s)+1}\right) \\
\theta_1 &= 2\pi f/f_s
\end{aligned}
$$

(b)

For the second filter,

$$
H(f) = H(z)|_{z=e^{j2\pi f/f_s}}
$$

$$
\begin{aligned}
&= \frac{(e^{j2\pi f/f_s} - 1)}{e^{j2\pi f/f_s}} \\
&= \frac{cos(2\pi f/f_s) + jsin(2\pi f/f_s) - 1}{e^{j2\pi f/f_s}}
\end{aligned}
$$

The magnitude response is

$$
|H(f)| = \sqrt{(cos(2\pi f/f_s) - 1)^2 + sin^2(2\pi f/f_s)}
$$

and phase response is

$$
arg(H(f)) = \arctan(\frac{sin(2\pi f/f_s)}{cos(2\pi f/f_s) - 1}) - (2\pi f/f_s)
$$

(c)

For the third filter,

$$
\begin{aligned}
H(f) &= H(z)|_{z=e^{j2\pi f/f_s}} \\
&= \frac{1}{e^{j2\pi f/f_s} - a} \\
&= \frac{1}{cos(2\pi f/f_s) + jsin(2\pi f/f_s) - a}
\end{aligned}
$$

The magnitude response is

$$
|H(f)| = \frac{1}{\sqrt{(cos(2\pi f/f_s) - a)^2 + sin^2(2\pi f/f_s)}}
$$

and phase response is

$$
arg(H(f)) = -\arctan(\frac{sin(2\pi f/f_s)}{cos(2\pi f/f_s) - a})
$$

Magnitude responses and phase responses calculated from Matlab are shown in Fig.3 and Fi.4.

We can observe that POI's results are the same as Matlab's results.

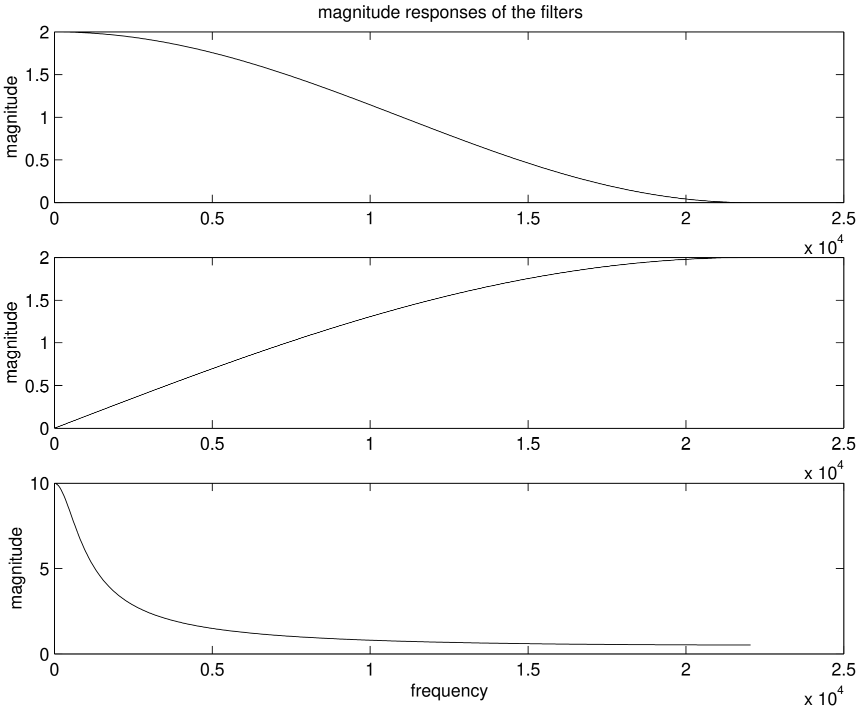

Fig. 3. Magnitude responses of the three filters obtained by Matlab

IV. Question 4

A linear time invariant filter is described by the difference equation $y[n] = 0.8y[n-1] - 0.8x[n] + x[n-1]$.

1) Find H(z) and identify the poles and zeros.

2) Find an expression for the frequency response, assuming $f_s = 44,100Hz$.

3) Show that the magnitude of the frequency response is 1 for all values of f.

Solution.

1) The z-transform of $y[n]$ is

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{\infty} y[k]z^{-k} \\ &= \sum_{k=-\infty}^{\infty} 0.8y[k-1]z^{-k} - 0.8x[k]z^{-k} + x[k-1]z^{-k} \\ &= 0.8z^{-1}Y(z) - 0.8X(z) + z^{-1}X(z) \end{aligned}$$

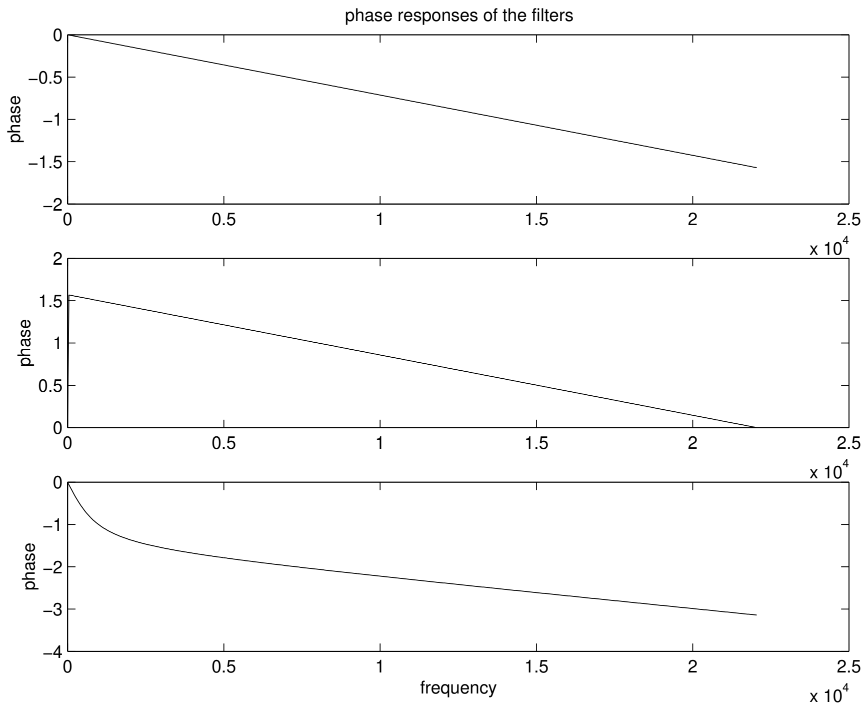

Fig. 4. Phase responses of the three filters obtained by Matlab

Thus

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{-0.8 + z^{-}1}{1 - 0.8z^{-}1} \\ &= -0.8\frac{z - 0.8^{-1}}{z - 0.8} \end{aligned}$$

$H(z)$ has pole at $z = 0.8$ and zero at $z = 0.8^{-1} = 1.25$.

2)

$$\begin{aligned} H(f) &= H(z)|_{z=e^{j\hat{\omega}}} \\ &= -0.8\frac{e^{j\hat{\omega}} - 0.8^{-1}}{e^{j\hat{\omega}} - 0.8} \end{aligned}$$

3)

$$|H(e^{j\hat{\omega}})|^2 = H(e^{j\hat{\omega}})H(e^{j\hat{\omega}})^*$$

$$
\begin{aligned}
&= (\frac{-0.8 + e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}})(\frac{-0.8 + e^{j\hat{\omega}}}{1 - 0.8e^{j\hat{\omega}}}) \\
&= \frac{-0.8e^{j\hat{\omega}} - 0.8e^{-j\hat{\omega}} + 1.64}{-0.8e^{j\hat{\omega}} - 0.8e^{-j\hat{\omega}} + 1.64} \\
&= 1
\end{aligned}
$$

Frequency response of the filter is shown in Fig.5.

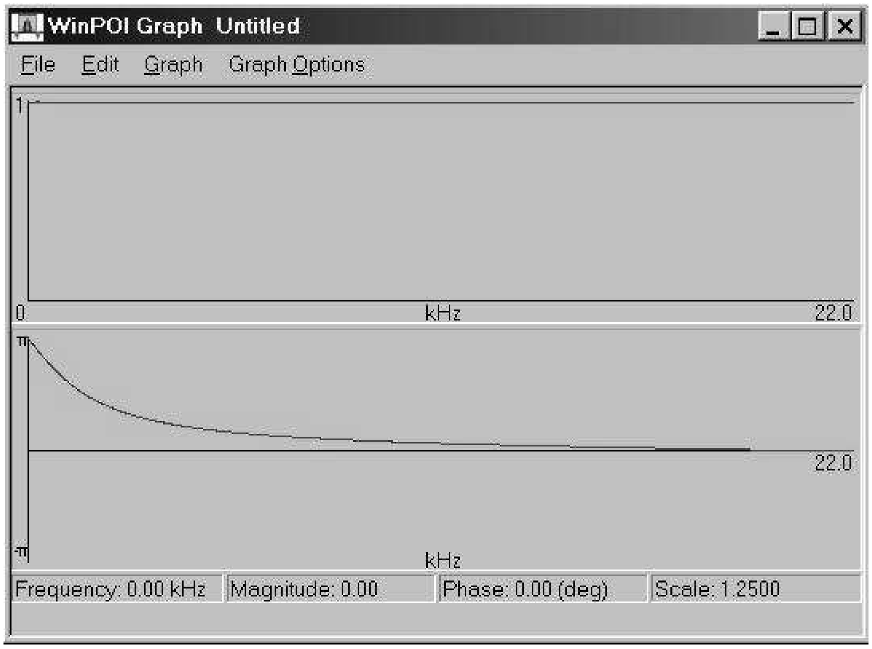

Fig. 5. Frequency response of H(z)