Solution to Assignment 4

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I. Question 1

1.1 For a transfer function with one pole and one zero, where (approximately) in the complex z-plane wolud you place the pole γ_1 and zero z_1 to obtain a high pass filter characteristic with zero response at $f = 0$ and maximum response at $f = f_s/2$?

1.2 Draw a sketch to show the pole and zero locations.

1.3 Draw a sketch of the approximate frequency (magnitude) response.

1.4 Derive the diference equation to implement this high pass filter in terms of γ_1 and z_1.

1.5 Draw a diagram of an impulse response of the filter.

Solution.

1.1

For a filter with one pole and one zero,

$$H[z] \quad = \quad b_0 \frac{z - z_1}{z - \gamma_1} \tag{1}$$

The frequency response becomes

$$\begin{aligned} H(f) &= H(z)_{z=e^{j\hat{\omega}}} && (2)\\ &= H(z)|_{z=e^{j2\pi f/f_s}} && (3)\\ &= b_0 \frac{z = e^{j2\pi f/f_s - z_1}}{z = e^{j2\pi f/f_s - \gamma_1}} && (4)\\ &= b_0 \frac{r_1}{d_1} e^{j(\phi_1 - \theta_1)} && (5) \end{aligned}$$

where

$$\begin{aligned} r_1 e^{j\phi_1} &= e^{j2\pi f/f_s} - z_1 && (6)\\ d_1 e^{j\theta_1} &= e^{j2\pi f/f_s} - \gamma_1 && (7) \end{aligned}$$

Assume $b_0 = 1$. From the table on the notes page 49, we can see that the pole γ_1 should be located on the negative real axis, whereas the zero z_1 on the positive real axis. If we leave the pole γ_1 on the negative real axis within the unit circle and move the zero z_1 to 1, $H(f)$ becomes zero at $f = 0$. For these z_1 and γ_1, $H(f)$ becomes maximum at $f = f_s/2$.

1.2

The pole and zero locations on the z-plane are shown in Fig.1(a).

1.3

Magnitude response of the H(f) is shown in Fig.1(b).

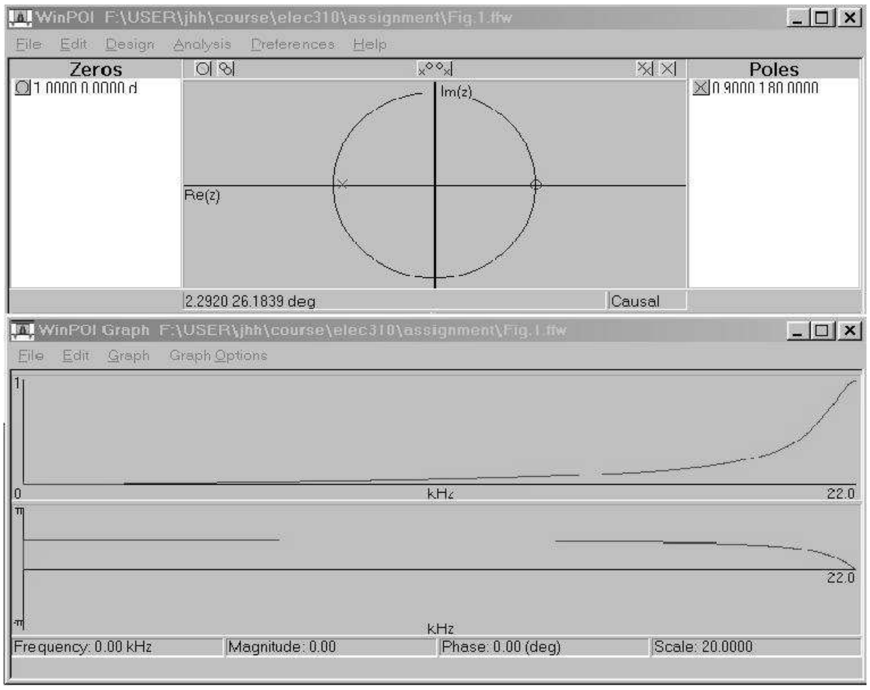

Fig. 1. Pole-zero locations and frequency response of the filter

1.4

From $H(z)$ in Eq.(1),

$$Y(z)(z-\gamma_1) = b_0 X(z)(z-z_1) \quad (8)$$

$$Y(z)(1-\gamma_1 z^{-1}) = b_0 X(z)(1-z_1 z^{-1}) \quad (9)$$

$$Y(z) = \gamma_1 z^{-1} Y(z) + b_0 X(z) - b_0 z_1 z^{-1} X(z) \quad (10)$$

$$(11)$$

By applying inverse z-transform,

$$y[n] = \gamma_1 y[n-1] + b_0 x[n] - b_0 z_1 x[n-1] \quad (12)$$

1.5

The block diagram of the filter is shown in Fig.2.

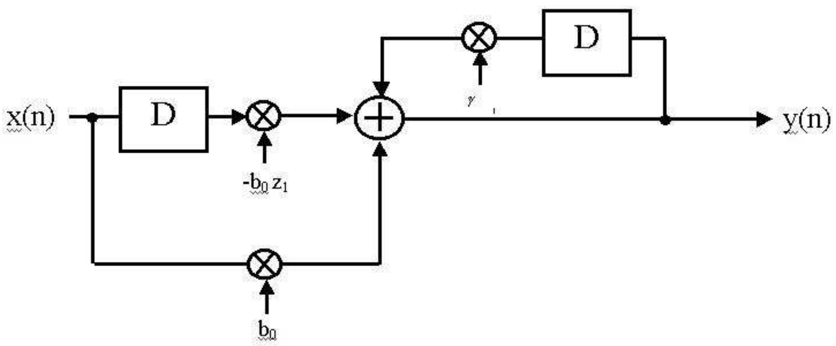

Fig. 2. Block diagram of the filter

II. Question 2

2.1 Find the (causal) inverse z-transform of

$$F[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)} \tag{13}$$

using partial fractions.

2.2 Write numerical values for the coefficients of the first two tersm.

Solution.

2.1

The Eq.(13) can be expressed as a factored form

$$F[z] = \frac{2z(3z+17)}{(z-1)(z-(3+j4))(z-(3-j4))} \tag{14}$$

First break $F(z)$ into partial fractions

$$\frac{F[z]}{z} = \frac{A}{z-1} + \frac{B}{z-(3+j4)} + \frac{C}{z-(3-j4)} \tag{15}$$

where

$$A = 2 \tag{16}$$

$$B = -1 - j\frac{5}{4} \tag{17}$$

$$C = -1 + j\frac{5}{4} \tag{18}$$

Now $F(z)$ is

$$F[z] = \frac{Az}{z-1} + \frac{Bz}{z-(3+j4)} + \frac{Cz}{z-(3-j4)} \tag{19}$$

So applying the z-transform pair

$$\frac{cz}{z-p} \leftrightarrow cp^k u[k] \tag{20}$$

We take the inverse z-transform to obtain $f[n]$

$$\begin{aligned} f[n] &= Au[n] + B(3+j4)^n u[n] + C(3-j4)^n u[n] && (21) \\ &= 2u[n] + (-1-j\frac{5}{4})(3+j4)^n u[n] + (-1+j\frac{5}{4})(3-j4)^n u[n] && (22) \end{aligned}$$

2.2

The numerical values for the first two terms are

$$\begin{aligned} n = 0 &: f[0] = 0 \\ n = 1 &: f[1] = 6 \end{aligned}$$

III. Question 3

Design a simple digital filter whose trasnfer function corresponds to a delay of two sample times, i.e.

$$y[k] = x[k-2] \tag{23}$$

Assume a sampling frequency $f_s = 48kHz$.

3.1 Specify $H(z)$.

3.2 Find the frequency response (amplitude and phase) of this filter.

3.3 Find the impulse response $h[k]$ of this filter by taking the inverse z-transform of $H(z)$.

3.4 Show by convolution that the output of the filter $y[k]$ is $x[k-2]$ as expected.

Solution.

3.1

Z-transform of $y[k]$ is

$$
\begin{aligned}
Y(z) &= \sum_{k=-\infty}^{\infty} y[k]z^{-k} && (24)\\
&= \sum_{k=-\infty}^{\infty} x[k-2]z^{-k} && (25)\\
&= z^{-2}X(z) && (26)
\end{aligned}
$$

Thus $H(z)$ is

$$
H(z) = Y(z)/X(z) = \frac{1}{z^2} \qquad (27)
$$

3.2

The frequency response $H(f)$ is

$$
\begin{aligned}
H(f) &= H(z)_{z=e^{j\hat{\omega}}} && (28)\\
&= H(z)|_{z=e^{j2\pi f/f_s}} && (29)\\
&= \frac{1}{(e^{j2\pi f/f_s})^2} && (30)\\
&= \frac{1}{e^{j4\pi f/f_s}} && (31)\\
&= \frac{1}{cos(4\pi f/f_s) + jsin(4\pi f/f_s)} && (32)\\
&= \frac{1}{d_1}e^{-j\theta_1} && (33)
\end{aligned}
$$

The magnitude response is

$$
|H(f)| = \frac{1}{d_1} \qquad (34)
$$

and phase response is

$$
arg(H(f)) = -\theta_1 \qquad (35)
$$

where

$$
\begin{aligned}
d_1 &= \sqrt{cos^2(4\pi f/f_s) + sin^2(4\pi f/f_s)} && (36)\\
\theta_1 &= 4\pi f/f_s && (37)
\end{aligned}
$$

3.3

From the z-transform pair

$$z^{-k} \leftrightarrow \delta[n-k] \tag{38}$$

The impulse response of the filter is

$$h[n] = \delta[n-2] \tag{39}$$

3.4

Given filter impulse response $h[k]$ and filter input sequence $x[n]$, the filter out is

$$\begin{aligned}
y[k] &= x[k] * h[k] && (40)\\
&= \sum_{n=0}^{\infty} h[n]x[k-n] && (41)\\
&= \sum_{n=0}^{\infty} \delta[n-2]x[k-n] && (42)\\
&= \delta[2-2]x[k-2] && (43)\\
&= x[k-2] && (44)
\end{aligned}$$