Solution to Assignment 5

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I. Question 1

Read Lathi section 5.3. Do exercise E5.9, page 371.

Solution.

Shift right by 2,

$$y[k] - \frac{5}{6}y[k-1] + \frac{1}{6}y[k-2] = 5f[k-1] - f[k-2] \tag{1}$$

Using unilateral z-transform,

$$\begin{aligned} y[k] &\Longrightarrow Y(z) && (2)\\ y[k-1] &\Longrightarrow \frac{1}{z}Y(z) + y[-1] && (3)\\ y[k-2] &\Longrightarrow \frac{1}{z^2}Y(z) + \frac{1}{z}y[-1] + y[-2] && (4)\\ f[k-1] &\Longrightarrow \frac{1}{z}F(z) + f[-1] && (5)\\ f[k-2] &\Longrightarrow \frac{1}{z^2}F(z) + \frac{1}{z}f[-1] + f[-2] && (6) \end{aligned}$$

z-transform of Eq.(1) is

$$\begin{aligned} Y(z) - \frac{5}{6}[\frac{1}{z}Y(z) + 2] + \frac{1}{6}[\frac{1}{z^2}Y(z) + \frac{2}{z}] &= 5[\frac{1}{z}\frac{z}{z-1}] - [\frac{1}{z^2}\frac{z}{z-1}] && (7)\\ (1 - \frac{5}{6}\frac{1}{z} + \frac{1}{6}\frac{1}{z^2})Y(z) - \underbrace{(\frac{10}{6} - \frac{2}{6z})}_{initial-condition-terms} &= \underbrace{\frac{5}{z-1} - \frac{1}{z(z-1)}}_{input-terms} && (8)\\ (z^2 - \frac{5}{6}z + \frac{1}{6})Y(z) &= \underbrace{z(\frac{5}{3}z - \frac{1}{3})}_{initial-condition-terms} + \underbrace{\frac{z(5z-1)}{z-1}}_{input-terms} && (9) \end{aligned}$$

Now Y(z) is

$$Y(z) = \underbrace{\frac{2z(5z-1)}{(2z-1)(3z-1)}}_{initial-condition-terms} + \underbrace{\frac{6z(5z-1)}{(2z-1)(3z-1)(z-1)}}_{input-terms} \tag{10}$$

Break Y(z) into partial fractions

$$\begin{aligned} \frac{Y(z)}{z} &= \frac{2(5z-1)}{(2z-1)(3z-1)} + \frac{6(5z-1)}{(2z-1)(3z-1)(z-1)} && (11)\\ &= \frac{A}{2z-1} + \frac{B}{3z-1} + \frac{C}{2z-1} + \frac{D}{z-1} + \frac{E}{3z-1} && (12) \end{aligned}$$

$$\begin{aligned} A &= 6 \\ B &= -4 \\ C &= -36 \\ D &= 12 \\ E &= 18 \end{aligned}$$

$$\frac{Y(z)}{z} = \frac{6}{2z-1} - \frac{4}{3z-1} - \frac{36}{2z-1} + \frac{12}{z-1} + \frac{18}{3z-1} \tag{13}$$

$$Y(z) = \frac{3z}{z-\frac{1}{2}} - \frac{\frac{4}{3}z}{z-\frac{1}{3}} - \frac{18z}{z-\frac{1}{2}} + \frac{12z}{z-1} + \frac{6z}{z-\frac{1}{3}} \tag{14}$$

Taking inverse z-transform

$$y[k] = \{ \underbrace{[3(\frac{1}{2})^k - \frac{4}{3}(\frac{1}{3})^k]}_{initial-condition-terms} + \underbrace{[12 - 18(\frac{1}{2})^k + 6(\frac{1}{3})^k]}_{input-terms} \}u[k] \tag{15}$$

$$= [12 - 15(\frac{1}{2})^k + \frac{14}{3}(\frac{1}{3})^k]u[k] \tag{16}$$

II. Question 2

Find $F_r = DFT\ \{f_k = [1\ 0\ 0\ 1]\}$ and plot the magnitude and phase of F_r. Find the $IDFT$ of F_r to verify it is equal to [1 0 0 1] as expected.

Solution.

The DFT of f_k is

$$F_r = \sum_{k=0}^{N_0-1} f_k (e^{j2\pi r/N_0})^{-k} \tag{17}$$

$$= 1 \cdot e^0 + 0 + 0 + 1 \cdot (e^{j2\pi r/N_0})^{-3} \tag{18}$$

$$= 1 + cos(3 \cdot 2\pi r/N_0) - j sin(3 \cdot 2\pi r/N_0) \tag{19}$$

The magnitude and phase responses are

$$|F_r| = \sqrt{(1 + cos(3 \cdot 2\pi r/N_0))^2 + sin^2(3 \cdot 2\pi r/N_0)} \tag{20}$$

$$arg(F_r) = arctan[\frac{-sin(3 \cdot 2\pi r/N_0)}{1 + cos(3 \cdot 2\pi r/N_0)}] \tag{21}$$

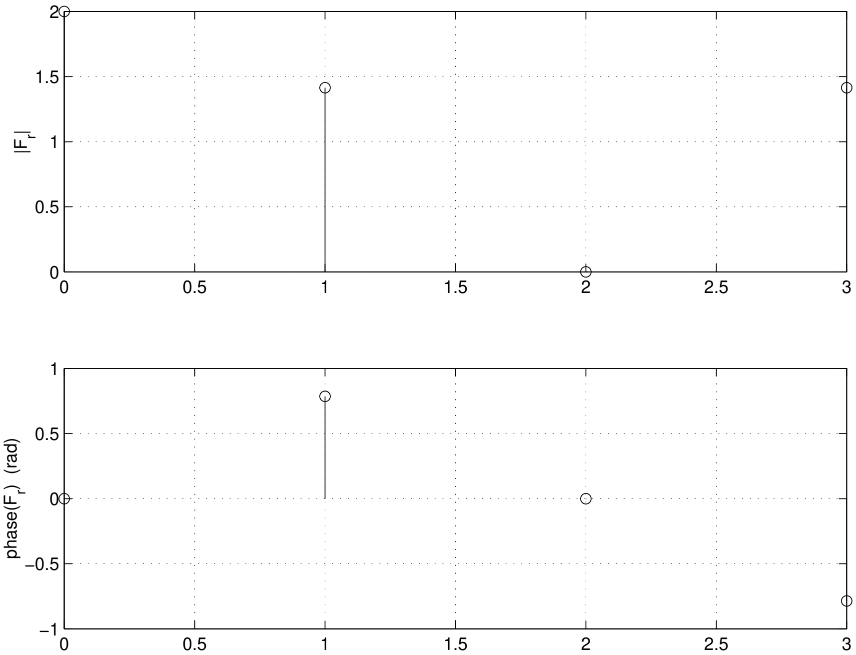

Fig. 1. Frequency responses of Fr

The IDFT of F_r is

$$f_k = \frac{1}{N_0}\sum_{r=0}^{N_0-1} F_r(e^{j2\pi r/N_0})^k \tag{22}$$

$$= \frac{1}{N_0}\sum_{r=0}^{N_0-1} [e^{-j2\pi r/N_0 \cdot 0} + e^{-j2\pi r/N_0 \cdot 3}]e^{j2\pi rk/N_0} \tag{23}$$

Using

$$\frac{1}{N_0}\sum_{r=0}^{N_0-1} e^{j2\pi rk/N_0}e^{-j2\pi rn/N_0} = \delta_{kn} \tag{24}$$

where

$$\delta_{kn} = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{otherwise} \end{cases}$$

f_k is

$$f_k = \delta_{k \cdot 0} + \delta_{k \cdot 3} \tag{25}$$

$$= \delta[k] + \delta[k-3] \tag{26}$$

or $f_k = [1\ 0\ 0\ 1]$