Improved Least-Squares Methods for Source Localization: An Iterative Re-Weighting Approach

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***Abstract*—Locating a radiating source from range or range-difference measurements in a passive sensor network has recently attracted an increasing amount of research interest as it finds applications in a wide range of network-based wireless systems. Striking results for these localization problems have emerged using squared range (SR-LS) or squared range-difference (SRD-LS) least-squares (LS) approaches. In this paper, we present improved LS methods that demonstrate improved localization performance when compared with the best known results from the literature.**

***Index Terms*—improved globally optimal solution; generalized trust region subproblems (GTRS); least squares; non-convex; iterative re-weighting; source localization; range measurements; range-difference measurements; Newton algorithm.**

I. Introduction

Least squares based algorithms constitute an important class of solution techniques as they are geometrically meaningful and often provide low complexity solution procedures with competitive estimation accuracy [1]-[12]. On the other hand, the error measure in an LS formulation for the localization problem of interest is shown to be highly non-convex, possessing multiple local solutions with degraded performance. This non-convexity excludes many local methods that are iterative, hence extremely sensitive to where the iteration begins. In the case of source localization, this inherent feature of local methods is particular problematic because the source location is assumed to be entirely unknown and can appear practically anywhere, thus the chances to secure a good initial point for a local algorithm are next to none. For these reasons, we are interested in *global* localization techniques that are either non-iterative or insensitive to initial iterate. One representative in the class of global localization methods is the convex-relaxation based algorithm for range measurements proposed in [8], where the LS model is relaxed to a semidefinite programming problem which is known to be convex [13]. Another representative in this class is reference [12], where localization problems for range as well as range difference measurements are addressed by developing solution methods for *squared* range LS (SR-LS) and *squared* range difference LS (SRD-LS) problems. The key new ingredient of the proposed algorithms is an iterative procedure where the SR-LS (or SRD-LS) algorithm is applied to a weighted sum of squared terms where the weights are carefully designed so that the iterates produced quickly converge to a solution which, on comparing with the best known results, is found to be considerably closer to the original range-based (or range-difference-based) LS solution that is known to be the maximum-likelihood estimator in the case of Gaussian white measurement noise. Numerical results are presented for performance evaluation and comparisons.

II. Source Localization From Range Measurements

A. Problem Statement

The source localization problem considered here involves a given array of m sensors specified by $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ where $\boldsymbol{a}_i \in R^n$ contains the n coordinates of the ith sensor in space R^n. Each sensor measures its distance to a radiating source $\boldsymbol{x} \in R^n$. Throughout it is assumed that only noisy copies of the distance data are available, hence the *range measurements* obey the model

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m. \tag{1}$$

where ε_i denotes the unknown noise that has occurred when the ith sensor measures its distance to source $\boldsymbol{x}$. Let $\boldsymbol{r} = [r_1\ r_2 \ldots r_m]^T$ and $\boldsymbol{\varepsilon} = [\varepsilon_1\ \varepsilon_2 \ldots \varepsilon_m]^T$, the source localization problem can be stated as to estimate the exact source location $\boldsymbol{x}$ from the noisy range measurements $\boldsymbol{r}$.

B. LS Formulations and Review of Related Work

For the localization problem at hand, the range-based least squares (R-LS) estimate refers to the solution of the problem

$$\underset{\boldsymbol{x}}{\text{minimize}}\ f(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \tag{2}$$

Formulation (2) is connected to the maximum-likelihood (ML) location estimation that determines $\boldsymbol{x}$ by examining the probabilistic model of the error vector $\boldsymbol{\varepsilon}$. If $\boldsymbol{\varepsilon}$ obeys a Gaussian distribution with zero mean and covariance $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, then the maximum likelihood (ML) location estimator in this case is known to be

$$\boldsymbol{x}_{ML} = \arg\min_{\boldsymbol{x} \in R^n} (\boldsymbol{r} - \boldsymbol{g})^T \Sigma^{-1} (\boldsymbol{r} - \boldsymbol{g}) \tag{3}$$

where $\boldsymbol{g} = [g_1\ g_2 \ldots\ g_m]^T$ with $g_i = \|\boldsymbol{x} - \boldsymbol{a}_i\|$. It follows immediately that the ML solution in (3) is identical to the R-LS solution of problem (2) when covariance $\boldsymbol{\Sigma}$ is proportional to the identity matrix, i.e., $\sigma_1^2 = \ldots = \sigma_m^2 = 1$.

There are many methods for continuous unconstrained optimization [14], however most of them are local methods in the sense they are sensitive to the choice of initial point, and give no guarantee to yield global solutions when applied to non-convex objective functions. Unfortunately, the objective function in (2) is highly non-convex, possessing many local minimizers even for small-scale systems.

Reference [8] addresses problem (2) by a convex relaxation technique where (2) is modified to a convex problem known as semidefinite programming [13]. A rather different approach is proposed in [12] where the localization problem (2) is tackled by developing techniques to address the *squared range based LS* (SR-LS) problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \sum_{i=1}^{m} (\|\boldsymbol{x}-\boldsymbol{a}_i\|^2 - r_i^2)^2 \tag{4}$$

whose global solution can be computed by converting (4) into the class of generalized trust region subproblems (GTRS) [15], [16] and exploring its KKT conditions which are both necessary and sufficient optimality conditions. Although SR-LS solves (4) exactly, the produced solution remains to be an approximation of the original LS problem in (2) because it is no longer a ML solution. The method, described in detail below, tries to reduce the gap between the two solutions.

C. An Iterative Re-Weighting Approach

1) Weighted squared range-based least squares formulation: We now consider a weighted SR-LS (WSR-LS) problem, namely,

$$\underset{\boldsymbol{x}}{\text{minimize}} \sum_{i=1}^{m} w_i \left(\|\boldsymbol{x}-\boldsymbol{a}_i\|^2 - r_i^2\right)^2 \tag{5}$$

Following [12], we convert (5) into a GTRS as

$$\underset{\boldsymbol{y}\in R^{n+1}}{\text{minimize}} \|\boldsymbol{A}_w\boldsymbol{y} - \boldsymbol{b}_w\|^2 \tag{6a}$$

$$\text{subject to: } \boldsymbol{y}^T\boldsymbol{D}\boldsymbol{y} + 2\boldsymbol{f}^T\boldsymbol{y} = 0 \tag{6b}$$

where $\boldsymbol{y} = [\boldsymbol{x}^T\alpha]^T$, $\alpha = \|\boldsymbol{x}\|$, $\boldsymbol{A}_w = \boldsymbol{\Gamma}\boldsymbol{A}$ and $\boldsymbol{b}_w = \boldsymbol{\Gamma}\boldsymbol{b}$ with $\boldsymbol{\Gamma} = \text{diag}\left(\sqrt{w_1}, \ldots, \sqrt{w_m}\right)$, and

$$\boldsymbol{A} = \begin{pmatrix} -2\boldsymbol{a}_1^T & 1 \\ \vdots & \vdots \\ -2\boldsymbol{a}_m^T & 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} r_1^T - \|\boldsymbol{a}_1\|^T \\ \vdots \\ r_m^T - \|\boldsymbol{a}_m\|^T \end{pmatrix} \tag{7}$$

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & 0 \end{pmatrix}, \boldsymbol{f} = \begin{pmatrix} \boldsymbol{0} \\ -0.5 \end{pmatrix} \tag{8}$$

For a *fixed* set of weights $\{w_i, i = 1, , m\}$, the global solution of (6) can be solved by the method developed in [12] and we denote the solution by $S(\boldsymbol{A}_w, \boldsymbol{b}_w, \boldsymbol{D}, \boldsymbol{f})$.

2) Moving the SR-LS solution towards R-LS solution via iterative re-weighting: The main idea here is to use the weights $\{w_i, i = 1, \ldots, m\}$ to tune the objective in (5) toward the objective in (2) so that the solution obtained by minimizing such a WSR-LS problem is expected to be closer toward that of problem (2). To substantiate the idea, we compare the ith term of the objective in (5) with its counterpart in (2), namely,

$$\underbrace{w_i\left(\|\boldsymbol{x}-\boldsymbol{a}_i\|^2 - r_i^2\right)^2}_{\text{in (5)}} \leftrightarrow \underbrace{\left(\|\boldsymbol{x}-\boldsymbol{a}_i\| - r_i\right)^2}_{\text{in (2)}} \tag{9}$$

and write the term in (5) as

$$\begin{aligned} &w_i\left(\|\boldsymbol{x}-\boldsymbol{a}_i\|^2 - r_i^2\right)^2 = \\ &w_i\left(\|\boldsymbol{x}-\boldsymbol{a}_i\| + r_i\right)^2 \underbrace{\left(\|\boldsymbol{x}-\boldsymbol{a}_i\| - r_i\right)^2}_{\text{same as in (2)}} \end{aligned} \tag{10}$$

It follows that the objective in (5) would be the same as in (2) if the weight w_i was assigned to $1/\left(\|\boldsymbol{x}-\boldsymbol{a}_i\| + r_i\right)^2$. Evidently, weight assignments as such cannot be realized because w_i's must be fixed constants for (5) to be a globally solvable WSR-LS problem. A natural remedy to deal with this technical difficulty is to employ an iterative procedure whose kth iteration generates a global solution $\boldsymbol{x}_k$ of a WSR-LS subproblem of the form

$$\underset{x}{\text{minimize}} \sum_{i=1}^{m} w_i^{(k)} \left(\|\boldsymbol{x}-\boldsymbol{a}_i\|^2 - r_i^2\right)^2 \tag{11}$$

where for $k \geq 2$ the weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ are assigned using the previous iterate $\boldsymbol{x}_{k-1}$ as

$$w_i^{(k)} = \frac{1}{\left(\|\boldsymbol{x}_{k-1}-\boldsymbol{a}_i\| + r_i\right)^2} \tag{12}$$

while for $k = 1$ all weights $\{w_i^{(1)}, i = 1, \ldots, m\}$ are set to unity. Clearly the weights given by (12) are realizable. More importantly, when the iterates produced by solving (12) (namely $\boldsymbol{x}_k$ for $k = 1, 2, \ldots$) converge, in the kth iteration with k sufficiently large, the objective function of (11) in a small vicinity of its solution $\boldsymbol{x}_k$ is approximately equal to

$$\begin{aligned} &\sum_{i=1}^{m} w_i^{(k)} \left(\|\boldsymbol{x}-\boldsymbol{a}_i\|^2 - r_i^2\right)^2 \approx \sum_{i=1}^{m} w_i^{(k)} \left(\|\boldsymbol{x}_k-\boldsymbol{a}_i\|^2 - r_i^2\right)^2 \\ &= \sum_{i=1}^{m} w_i^{(k)} \left(\|\boldsymbol{x}_k-\boldsymbol{a}_i\| + r_i\right)^2 \left(\|\boldsymbol{x}_k-\boldsymbol{a}_i\| - r_i\right)^2 \\ &\approx \sum_{i=1}^{m} w_i^{(k)} \left(\|\boldsymbol{x}_{k-1}-\boldsymbol{a}_i\| + r_i\right)^2 \left(\|\boldsymbol{x}_k-\boldsymbol{a}_i\| - r_i\right)^2 \\ &= \sum_{i=1}^{m} \left(\|\boldsymbol{x}_k-\boldsymbol{a}_i\| - r_i\right)^2 \approx \sum_{i=1}^{m} \left(\|\boldsymbol{x}-\boldsymbol{a}_i\| - r_i\right)^2 \end{aligned}$$

In words, with the weights from (12) the iterates produced by iteratively solving WSR-LS problem (11) are expected to approach the global solution of (2). The localization method can be outlined as follows.

Algorithm 1

1) Input data: Sensor locations $\{\boldsymbol{a}_i, i = 1, \ldots, m\}$, range measurements $\{r_i, i = 1, \ldots, m\}$, maximum number of iterations k_{max} and convergence tolerance ζ.

2) Generate data set $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{f}$ using (7) and (8). Set $k = 1, w_i^{(1)} = 1$ for $i = 1, \ldots, m$.

3) Set $\boldsymbol{\Gamma}_k = \text{diag}\left(\sqrt{w_1^{(k)}}, \ldots, \sqrt{w_m^{(k)}}\right)$, $\boldsymbol{A}_w = \boldsymbol{\Gamma}_k \boldsymbol{A}$ and $\boldsymbol{b}_w = \boldsymbol{\Gamma}_k \boldsymbol{b}$.

4) Solve the WSR-LS problem (11) via (6) to obtain its global solution $\boldsymbol{x}_k = S(\boldsymbol{A}_w, \boldsymbol{b}_w, \boldsymbol{D}, \boldsymbol{f})$.

5) If $k = k_{max}$ or $\|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\| < \zeta$, terminate and output $\boldsymbol{x}_k$ as the solution; otherwise, set $k = k + 1$, update weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ using (12), and repeat from Step 3).

It is evident that the complexity of the algorithm is practically equal to the complexity of the WSR-LS solver involved in Step 4 times the number of iterations, k. The algorithm converges with a small number of iterations, typically a $k \leq 6$ suffices. Convergence is guaranteed due to the nature of the WSR-LS solver which produces a global solution that does not depend on the initial point, and convexity of the WSR-LS problem (6). For simplicity, we shall call the solutions obtained from Algorithm 1 IRWSR-LS solutions.

III. Source Localization From Range-Difference Measurements

A. Problem Formulation

Another type of source localization problem that has attracted considerable attention is that of localizing a radiating source using range-difference measurements [11], [12]. In practice, range-difference measurements may be obtained by comparing the signal as it is received at the $m + 1$ sensors taken in pairs. As usual, these sensors are denoted as $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \in R^n$ with $\boldsymbol{a}_0$ be placed at the origin and used as a *reference sensor*. The range-difference measurements are obtained as

$$d_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x} - \boldsymbol{a}_0\| = \|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x}\|, \quad i = 1, \ldots, m \tag{13}$$

and the problem here is to estimate the location of a radiating source $\boldsymbol{x}$ based on measurements d_i's. Therefore, the standard range-difference LS (RD-LS) problem is formulated as

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}}\, F(\boldsymbol{x}) = \sum_{i=1}^{m} \left(d_i + \|\boldsymbol{x}\| - \|\boldsymbol{x} - \boldsymbol{a}_i\|\right)^2 \tag{14}$$

Reference [12] proposes a squared range-difference LS (SRD-LS) approach to obtain an approximate solution of (14), which is summarized below. By writing (13) as $d_i + \|\boldsymbol{x}\| = \|\boldsymbol{x} - \boldsymbol{a}_i\|$ and squaring both sides, we obtain

$$\left(d_i + \|\boldsymbol{x}\|\right)^2 = \|\boldsymbol{x} - \boldsymbol{a}_i\|^2 \tag{15}$$

which can be simplified to

$$-2d_i\|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T\boldsymbol{x} = g_i, \quad i = 1, \ldots, m \tag{16}$$

where $g_i = d_i^2 - \|\boldsymbol{a}_i\|^2$. If d_i's in (16) are taken to be real-world data, then we only have

$$-2d_i\|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T\boldsymbol{x} - g_i \approx 0, \quad i = 1, \ldots, m \tag{17}$$

Reference [12] proposes an LS solution for the problem at hand by minimizing the sum of squared residues on the left side of (17), namely,

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}} \sum_{i=1}^{m} \left(-2\boldsymbol{a}_i^T\boldsymbol{x} - 2d_i\|\boldsymbol{x}\| - g_i\right)^2 \tag{18}$$

By introducing new variable $\boldsymbol{y} = [\boldsymbol{x}^T \|\boldsymbol{x}\|]^T$ and noticing non-negativity of the component y_{n+1} problem (18) is converted to

$$\underset{\boldsymbol{y} \in R^{n+1}}{\text{minimize}}\, \|\boldsymbol{B}\boldsymbol{y} - \boldsymbol{g}\|^2 \tag{19a}$$

$$\text{subject to: } \boldsymbol{y}^T\boldsymbol{C}\boldsymbol{y} = 0 \tag{19b}$$

$$y_{n+1} \geq 0 \tag{19c}$$

where $\boldsymbol{g} = [g_1 \ldots g_m]^T$ and

$$\boldsymbol{B} = \begin{pmatrix} -2\boldsymbol{a}_1^T & -2d_1 \\ \vdots & \vdots \\ -2\boldsymbol{a}_m^T & -2d_m \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{1\times n} & -1 \end{pmatrix} \tag{20}$$

Reference [12] presents a rigorous argument which shows that the optimal solution of (19) either assumes the form of $\tilde{\boldsymbol{y}}(\lambda) = \left(\boldsymbol{B}^T\boldsymbol{B} + \lambda\boldsymbol{C}\right)^{-1}\boldsymbol{B}^T\boldsymbol{g}$ where λ solves

$$\tilde{\boldsymbol{y}}(\lambda)^T\boldsymbol{C}\tilde{\boldsymbol{y}}(\lambda) = 0 \tag{21}$$

and makes $\boldsymbol{B}^T\boldsymbol{B} + \lambda\boldsymbol{C}$ positive definite, or is the vector among $\{\boldsymbol{0}, \tilde{\boldsymbol{y}}(\lambda_1), \ldots, \tilde{\boldsymbol{y}}(\lambda_p)\}$ that gives the smallest objective function in (19a), where $\{\lambda_i, i = 1, \ldots, p\}$ are all roots of (21) such that the $(n+1)$'th component of $\tilde{\boldsymbol{y}}(\lambda_i)$ is nonnegative and $\boldsymbol{B}^T\boldsymbol{B} + \lambda\boldsymbol{C}$ has exactly one negative and n positive eigenvalues. We shall refer the global solution of (19) to as the SRD-LS solution.

B. Improved Solution Using Iterative Re-weighting

1) The Algorithm: We now present a method for improved solutions over SRD-LS solutions. The method incorporates an iterative re-weighting procedure into the SRD-LS approach, hence it is in spirit similar to the IRWRS-LS approach described in Sec. 2.3.2. We begin by considering the weighted SRD-LS problem

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}} \sum_{i=1}^{m} w_i \left(-2\boldsymbol{a}_i^T\boldsymbol{x} - 2d_i\|\boldsymbol{x}\| - g_i\right)^2 \tag{22}$$

where weights w_i for $i = 1, \ldots, m$ are fixed nonnegative constants. The counterpart of (19) for the problem (22) is given by

$$\underset{\boldsymbol{y} \in R^{n+1}}{\text{minimize}}\, \|\boldsymbol{B}_w\boldsymbol{y} - \boldsymbol{g}_w\| \tag{23a}$$

$$\text{subject to: } \boldsymbol{y}^T\boldsymbol{C}\boldsymbol{y} = 0 \tag{23b}$$

$$y_{n+1} \geq 0 \tag{23c}$$

where $\boldsymbol{B}_w = \boldsymbol{\Gamma}\boldsymbol{B}$, $\boldsymbol{g}_w = \boldsymbol{\Gamma}\boldsymbol{g}$ and $\boldsymbol{\Gamma} = \text{diag}\{\sqrt{w_1}, \ldots, \sqrt{w_m}\}$, which will be referred to as the weighted SRD-LS (WSRD-LS) problem. On comparing (23) with (19), it follows immediately that the global solver for problem (19) characterized by data set $\{\boldsymbol{B}, \boldsymbol{g}, \boldsymbol{C}\}$ can also be used for solving problem (23) by applying it to data set $\{\boldsymbol{B}_w, \boldsymbol{g}_w, \boldsymbol{C}\}$.

Concerning the assignment of weights $\{w_i, i = 1, \ldots, w_m\}$, we recall (15), (16) and observe that the ith term of the objective function in (22) can be written as

$$\begin{aligned} & w_i \left(-2d_i\|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T\boldsymbol{x} - g_i\right)^2 \\ =& w_i \left((d_i + \|\boldsymbol{x}\|)^2 - \|\boldsymbol{x} - \boldsymbol{a}_i\|^2\right)^2 \\ =& w_i \left(d_i + \|\boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}_i\|\right)\left(d_i + \|\boldsymbol{x}\| - \|\boldsymbol{x} - \boldsymbol{a}_i\|\right) \end{aligned}$$

Clearly, the last expression above would become the ith term of the objective function in the RD-LS problem (14) if weights w_i were set to $1/\left(d_i + \|\boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}_i\|\right)^2$ so that the first two factors are cancelled out. This suggests that a realizable weight assignment for performing practically the same cancellation can be made by means of iterative re-weighting for problems (22) and (23) where the weights in the kth iteration are assigned to

$$w_i^{(k)} = \frac{1}{\left(d_i + \|\boldsymbol{x}_{k-1}\| + \|\boldsymbol{x}_{k-1} - \boldsymbol{a}_i\|\right)^2}, i = 1, \ldots, m \quad (24)$$

Based on the analysis above, a localization algorithm for range-difference measurements can be outlined as follows.

Algorithm 2

1) Input data: Sensor locations $\{\boldsymbol{a}_i, i = 0, 1, \ldots, m\}$ with $\boldsymbol{a}_0 = \boldsymbol{0}$, range-difference measurements $\{d_i, i = 1, \ldots, m\}$, maximum number of iterations k_{max} and convergence tolerance ξ.

2) Generate data set $\{\boldsymbol{B}, \boldsymbol{g}, \boldsymbol{C}\}$ using (20). Set $k = 1$, $w_i^{(1)} = 1$ for $i = 1, \ldots, m$.

3) Set $\boldsymbol{\Gamma}_k = \text{diag}\left(\sqrt{w_1^{(k)}}, \ldots, \sqrt{w_m^{(k)}}\right)$, $\boldsymbol{B}_w = \boldsymbol{\Gamma}_k\boldsymbol{B}$ and $\boldsymbol{g}_w = \boldsymbol{\Gamma}_k\boldsymbol{g}$.

4) Solve WSRD-LS problem (23) to obtain its global solution $\boldsymbol{x}_k$.

5) If $k = k_{max}$ or $\|\boldsymbol{x}_k - \boldsymbol{x}_{k-1}\| < \xi$, terminate and output $\boldsymbol{x}_k$ as the solution; otherwise, set $k = k + 1$, update weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ using (24c), and repeat from Step 3). We shall call the solutions obtained from Algorithm 2 IRWSRD-LS solutions.

IV. Numerical Results

Performance of the proposed algorithms was evaluated and compared with existing state-of-the-art methods by Monte Carlo simulations with a set-up similar to that of [12]. SR-LS and SRD-LS solutions were used as performance benchmarks for Algorithm 1 and 2, respectively. In both cases the system consisted of m sensors $\{\boldsymbol{a}_i, i = 1, 2, \ldots, m\}$ randomly placed in the planar region in R^2, and a radiating source $\boldsymbol{x}_s$, located randomly in the region $\{\boldsymbol{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$. The coordinates of the source and sensors were generated for each dimension following a uniform distribution. Measurement noise $\{\varepsilon_i, i = 1, \ldots, m\}$ was modelled as independent and identically distributed (i.i.d) random variables with zero mean and variance σ^2. Accuracy of source location estimation was evaluated in terms of mean squared error in the form MSE $= E\{\|\boldsymbol{x}^* - \boldsymbol{x}_s\|\}$ where $\boldsymbol{x}_s$ denotes the exact source location and $\boldsymbol{x}^*$ is its estimation obtained by SR-LS(SRD-LS) and IRWSR-LS(IRWSRD-LS) methods, respectively. Table I and II provide comparisons of these methods with SR-LS(SRD-LS), where each entry was averaged MSE over 10,000 Monte Carlo runs of the method. The last column in each table represents relative improvement of the proposed methods over SR-LS(SRD-LS) solutions in percentage.

TABLE I
Averaged MSE for SR-LS and IRWSR-LS methods

σ	SR - LS	IRWSR-LS	Improvement, %
1e-03	1.897294e-06	1.123411e-06	40.8
1e-02	1.779870e-04	1.081470e-04	39.2
1e-01	1.831870e-02	1.128165e-02	38.4
1e+0	2.415438e+00	1.877930e+00	22.3

TABLE II
Averaged MSE for SRD-LS and IRWSRD-LS methods

σ	SRD - LS	IRWSRD-LS	Improvement, %
1e-04	8.4918e-09	4.1050e-09	51.7
1e-03	5.8553e-06	3.5105e-06	40.0
1e-02	6.3508e-05	5.0378e-05	20.7
1e-01	1.6057e-02	1.0055e-02	37.3
1e+0	1.2773e+00	6.2221e-01	51.2

Employing a set-up similar to that in [12], the simulation studies of Algorithm 1 considered $m = 5$ sensors placed in the region $[-15; 15] \times [-15; 15]$, with σ being one of four possible levels $\{10^{-3}, 10^{-2}, 10^{-1}, 1\}$. The range measurements $\{r_i, i = 1, 2, \ldots, 5\}$ were calculated using (1) and Step 4 of Algorithm 1 was implemented using the SR-LS algorithm proposed in [12]. It is observed that IRWSR-LS solutions offer considerable improvement over SR-LS solutions.

The simulation studies of Algorithm 2 considered $m = 11$ sensors $\{\boldsymbol{a}_i, i = 1, 2, \ldots, 10\}$ with $\boldsymbol{a}_0 = \boldsymbol{0}$ and other ten sensors placed in the region $[-15; 15] \times [-15; 15]$, with σ being one of five possible levels $\{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1\}$. The range-difference measurements used in matrix $\boldsymbol{B}$ in (20) were calculated as a noise-contaminated version of d_i in (13) plus ε_i, and Step 4 of Algorithm 2 was carried out using the SRD-LS algorithm in [12]. Again, the IRWSRD-LS solutions offer considerable improvement over SRD-LS solutions.

V. Conclusions

In this paper, new global methods for locating a radiating source based on noisy range or range difference measurements have been proposed. These methods are developed by transforming the SR-LS and SRD-LS algorithms [12] into an iterative procedure where adaptively weighted SR-LS (SRD-LS) objective is minimized so as to move an SR-LS (SRD-LS) objective towards the original R-LS objective as the iteration continues. Numerical results are presented to illustrate the proposed algorithms in comparison with the best known results from the literature.

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