Realization with Minimal Weighted Pole and Zero Sensitivity Subject to l_2-Scaling Constraints for Recursive Digital Filters

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***Abstract*— This paper deals with the problem of minimizing weighted pole and zero sensitivity subject to l_2-scaling constraints for state-space digital filters. A measure for pole and zero sensitivity is presented and an efficient iterative technique for minimizing this measure subject to l_2-scaling constraints is developed by relaxing the constraints into a single constraint on matrix trace and solving the relaxed problem with an effective matrix iteration scheme. A numerical example is included to demonstrate the validity and effectiveness of the proposed technique.**

I. INTRODUCTION

When a transfer function with infinite accuracy coefficients is realized by a state-space model with a finite binary representation, the state-space parameters must be truncated or rounded to fit the finite-word-length (FWL) constraints. This coefficient quantization usually alters the characteristics of the filter. For instance, it may change a stable filter to unstable one. This motivates the study of minimizing coefficients sensitivity. In this connection, the pole and zero sensitivity of a filter with respect to state-space parameters has been analyzed, and its reduction or minimization have been considered [1]-[6]. It has been mentioned in [3] that the sensitivities of the poles and zeros of a transfer function with respect to the variations in state-space parameters (due to FWL) are closely related to the sensitivity characteristics of its frequency transfer function. Although minimal pole sensitivity is achieved when the system matrix $\boldsymbol{A}$ is normal, and minimal zero sensitivity is achieved when a certain matrix is normal, minimal pole sensitivity does not guarantee minimal zero sensitivity and vice versa. In [5], a method for minimizing a zero sensitivity measure subject to minimal pole sensitivity for state-space digital filters has been explored, since it is generally impossible to minimize pole sensitivity and zero sensitivity simultaneously. However, l_2-scaling for preventing overflow oscillations was not taken into account in [5]. Recently, the minimization problem of weighted pole and zero sensitivity was examined and a quasi-Newton algorithm was applied to minimize the weighted pole and zero sensitivity in [6], however l_2-scaling constraints were not imposed.

This paper investigates the problem of minimizing weighted pole and zero sensitivity subject to l_2-scaling constraints for state-space digital filters. An efficient iterative technique for minimizing a weighted pole and zero sensitivity measure subject to l_2-scaling constraints is developed by relaxing the constraints into a single constraint on matrix trace and solving the relaxed problem with an effective matrix iteration scheme. A numerical example is presented to demonstrate the validity and effectiveness of the proposed technique.

II. PROBLEM FORMULATION

A. Weighted Pole and Zero Sensitivity Measure

Consider a stable, controllable and observable state-space digital filter $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)_n$ of order n described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) \\ y(k) &= \boldsymbol{c}\boldsymbol{x}(k) + du(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k)$ is an $n \times 1$ state-variable vector, $u(k)$ is a scalar input, $y(k)$ is a scalar output, and $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ and d are real constant matrices of appropriate dimensions. The transfer function of the digital filter in (1) can be expressed as

$$H(z) = \boldsymbol{c}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b} + d. \tag{2}$$

The poles $\{\lambda_l\}$ of $H(z)$ are denoted by $\{\lambda_l\} = \lambda(\boldsymbol{A})$, while the zeros $\{v_l\}$ of $H(z)$ are indicated by $\{v_l\} = \lambda(\boldsymbol{Z})$ with

$$\boldsymbol{Z} = \boldsymbol{A} - d^{-1}\boldsymbol{b}\boldsymbol{c} \tag{3}$$

provided that $d \neq 0$ [7]. Hereafter, it is assumed that a direct path from the input to the output always exists in (1), i.e., $d \neq 0$.

Let the pole sensitivity matrix for the lth eigenvalue λ_l of matrix $\boldsymbol{A}$ be defined by [2]

$$\frac{\partial \lambda_l}{\partial \boldsymbol{A}} = \begin{bmatrix} \dfrac{\partial \lambda_l}{\partial a_{11}} & \cdots & \dfrac{\partial \lambda_l}{\partial a_{1n}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \lambda_l}{\partial a_{n1}} & \cdots & \dfrac{\partial \lambda_l}{\partial a_{nn}} \end{bmatrix}. \tag{4}$$

where a_{ij} for $i, j = 1, 2, \cdots, n$ denote the (i, j)th element of matrix $\boldsymbol{A}$. It is known that [2], [5]

$$\left(\frac{\partial \lambda_l}{\partial \boldsymbol{A}}\right)^T = \boldsymbol{x}_p(l)\boldsymbol{y}_p^H(l) \quad \text{for} \quad l = 1, 2, \cdots, n \tag{5}$$

holds provided that $\boldsymbol{A}$ has a full set of linearly independent eigenvectors, where $\boldsymbol{x}_p(l)$ is a right eigenvector corresponding to λ_l and $\boldsymbol{y}_p(l)$ is the reciprocal left eigenvector corresponding to $\boldsymbol{x}_p(l)$. Similarly, if $\boldsymbol{Z}$ has a linearly independent eigenvector set, then we have

$$\left(\frac{\partial v_l}{\partial \boldsymbol{Z}}\right)^T = \boldsymbol{x}_z(l)\boldsymbol{y}_z^H(l) \quad \text{for} \quad l = 1, 2, \cdots, n \tag{6}$$

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where $\boldsymbol{x}_z(l)$ is a right eigenvector corresponding to v_l and $\boldsymbol{y}_z(l)$ is the reciprocal left eigenvector corresponding to $\boldsymbol{x}_z(l)$. Moreover, it is known that [4]

$$\begin{aligned}\frac{\partial v_l}{\partial \boldsymbol{A}} &= \frac{\partial v_l}{\partial \boldsymbol{Z}}, & \frac{\partial v_l}{\partial \boldsymbol{b}} &= -d^{-1}\frac{\partial v_l}{\partial \boldsymbol{Z}}\boldsymbol{c}^T \\ \frac{\partial v_l}{\partial \boldsymbol{c}} &= -d^{-1}\boldsymbol{b}^T\frac{\partial v_l}{\partial \boldsymbol{Z}}, & \frac{\partial v_l}{\partial d} &= d^{-2}\boldsymbol{b}^T\frac{\partial v_l}{\partial \boldsymbol{Z}}\boldsymbol{c}^T\end{aligned} \tag{7}$$

are satisfied.

Using the Frobenius norm, the pole sensitivity measure J_p and the zero sensitivity measure J_z for the digital filter in (1) can be defined as [5]

$$\begin{aligned}J_p &= \sum_{l=1}^{n}\left\|\frac{\partial \lambda_l}{\partial \boldsymbol{A}}\right\|_F^2 = \sum_{l=1}^{n}\left\|\Big(\frac{\partial \lambda_l}{\partial \boldsymbol{A}}\Big)^T\right\|_F^2 \\ J_z &= \sum_{l=1}^{n}\Big\{\left\|\frac{\partial v_l}{\partial \boldsymbol{A}}\right\|_F^2 + \left\|\frac{\partial v_l}{\partial \boldsymbol{b}}\right\|_F^2 + \left\|\frac{\partial v_l}{\partial \boldsymbol{c}}\right\|_F^2 + \left\|\frac{\partial v_l}{\partial d}\right\|_F^2\Big\}\end{aligned} \tag{8}$$

respectively. By virtue of (5), (6) and (7), we can write (8) as

$$J_p = \sum_{l=1}^{n}\Big(\boldsymbol{x}_p^H(l)\boldsymbol{x}_p(l)\Big)\Big(\boldsymbol{y}_p^H(l)\boldsymbol{y}_p(l)\Big) \tag{9a}$$

$$J_z = \sum_{l=1}^{n}\Big(\boldsymbol{x}_z^H(l)\boldsymbol{x}_z(l) + \alpha_l^2\Big)\Big(\boldsymbol{y}_z^H(l)\boldsymbol{y}_z(l) + \beta_l^2\Big) \tag{9b}$$

where

$$\alpha_l = |d^{-1}\boldsymbol{c}\boldsymbol{x}_z(l)|, \qquad \beta_l = |d^{-1}\boldsymbol{y}_z^H(l)\boldsymbol{b}|. \tag{9c}$$

If a coordinate transformation defined by

$$\overline{\boldsymbol{x}}(k) = \boldsymbol{T}^{-1}\boldsymbol{x}(k) \tag{10}$$

is applied to the digital filter in (1), then the new realization associated with state variable $\overline{\boldsymbol{x}}(k)$ can be characterized by $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{c}}, d)_n$ where

$$\overline{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \quad \overline{\boldsymbol{b}} = \boldsymbol{T}^{-1}\boldsymbol{b}, \quad \overline{\boldsymbol{c}} = \boldsymbol{c}\boldsymbol{T}. \tag{11}$$

For this realization, the right eigenvector $\boldsymbol{x}_p(l)$ corresponding to λ_l and the reciprocal left eigenvector $\boldsymbol{y}_p(l)$ corresponding to $\boldsymbol{x}_p(l)$ are changed to [3], [5]

$$\overline{\boldsymbol{x}}_p(l) = \boldsymbol{T}^{-1}\boldsymbol{x}_p(l), \qquad \overline{\boldsymbol{y}}_p(l) = \boldsymbol{T}^T\boldsymbol{y}_p(l), \tag{12}$$

respectively, for $l = 1, 2, \cdots, n$. Similarly, the right eigenvector $\boldsymbol{x}_z(l)$ corresponding to v_l and the reciprocal left eigenvector $\boldsymbol{y}_z(l)$ corresponding to $\boldsymbol{x}_z(l)$ are changed to

$$\overline{\boldsymbol{x}}_z(l) = \boldsymbol{T}^{-1}\boldsymbol{x}_z(l), \qquad \overline{\boldsymbol{y}}_z(l) = \boldsymbol{T}^T\boldsymbol{y}_z(l), \tag{13}$$

respectively, for $l = 1, 2, \cdots, n$. Therefore, for the realization $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{c}}, d)_n$ specified in (11), the pole and zero sensitivity measures in (9a) and (9b) can be expressed as

$$\begin{aligned}J_p(\boldsymbol{P}) &= \sum_{l=1}^{n}\Big(\boldsymbol{x}_p^H(l)\boldsymbol{P}^{-1}\boldsymbol{x}_p(l)\Big)\Big(\boldsymbol{y}_p^H(l)\boldsymbol{P}\boldsymbol{y}_p(l)\Big) \\ J_z(\boldsymbol{P}) &= \sum_{l=1}^{n}\Big(\boldsymbol{x}_z^H(l)\boldsymbol{P}^{-1}\boldsymbol{x}_z(l) + \alpha_l^2\Big)\Big(\boldsymbol{y}_z^H(l)\boldsymbol{P}\boldsymbol{y}_z(l) + \beta_l^2\Big)\end{aligned} \tag{14}$$

respectively, where $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$. It is noted that for any $n \times n$ nonsingular matrix $\boldsymbol{T}$

$$J_p(\boldsymbol{P}) \ge n, \qquad J_z(\boldsymbol{P}) \ge \sum_{l=1}^{n}(1 + |\alpha_l\beta_l|)^2 \tag{15}$$

are always hold true [2]-[5]. In this paper, we consider a weighted pole and zero sensitivity measure $J_\gamma(\boldsymbol{P})$ defined by

$$J_\gamma(\boldsymbol{P}) = \gamma J_p(\boldsymbol{P}) + (1-\gamma)J_z(\boldsymbol{P}) \tag{16}$$

where $0 \le \gamma \le 1$ is a weighting factor. Our consideration of a weighted measure as in (16) is motivated by the fact that it allows to study various combined pole and zero sensitivity to its parameters by assigning weight γ to various values, with the extreme circumstances of "minimizing pole sensitivity only" and "minimizing zero sensitivity only" as two special cases of $\gamma = 1$ and $\gamma = 0$, respectively.

B. l_2-Scaling Constraints and Problem Statement

The controllability Gramian $\boldsymbol{K}_c$ of the digital filter in (1) plays an important role in the dynamic-range scaling of the state-variable vector $\overline{\boldsymbol{x}}(k)$, and matrix $\boldsymbol{K}_c$ can be obtained by solving the Lyapunov equation

$$\boldsymbol{K}_c = \boldsymbol{A}\boldsymbol{K}_c\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T. \tag{17}$$

With an equivalent realization as specified in (11), the controllability Gramian assumes the form

$$\overline{\boldsymbol{K}}_c = \boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T}. \tag{18}$$

If l_2-scaling constraints are imposed on the new state-variable vector $\overline{\boldsymbol{x}}(k)$ in (10), it is required that

$$(\overline{\boldsymbol{K}}_c)_{ii} = (\boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T})_{ii} = 1 \ \text{ for } \ i = 1, 2, \cdots, n. \tag{19}$$

The problem being considered in this paper is to obtain an $n \times n$ coordinate transformation matrix $\boldsymbol{T}$ that minimizes (16) subject to the l_2-scaling constraints in (19).

III. POLE AND ZERO SENSITIVITY MINIMIZATION SUBJECT TO l_2-SCALING CONSTRAINTS

A. Minimization of (16) Subject to l_2-Scaling Constraints

In order to minimize (16) with $J_p(\boldsymbol{P})$ and $J_z(\boldsymbol{P})$ given by (14) with respect to an $n \times n$ symmetric positive-definite matrix $\boldsymbol{P}$ subject to l_2-scaling constraints in (19), we define the Lagrange function

$$I_\gamma(\boldsymbol{P}, \xi) = J_\gamma(\boldsymbol{P}) + \xi\big(\mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] - n\big) \tag{20}$$

where ξ is a Lagrange multiplier, and compute the gradients

$$\begin{aligned}\frac{\partial I_\gamma(\boldsymbol{P}, \xi)}{\partial \boldsymbol{P}} &= \boldsymbol{M}_\gamma(\boldsymbol{P}) - \boldsymbol{P}^{-1}\boldsymbol{N}_\gamma(\boldsymbol{P})\boldsymbol{P}^{-1} - \xi\boldsymbol{P}^{-1}\boldsymbol{K}_c\boldsymbol{P}^{-1} \\ \frac{\partial I_\gamma(\boldsymbol{P}, \xi)}{\partial \xi} &= \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] - n\end{aligned} \tag{21}$$

where

$$\boldsymbol{M}_\gamma(\boldsymbol{P}) = \gamma \sum_{l=1}^{n} \mathrm{tr}\big[\boldsymbol{x}_p(l)\boldsymbol{x}_p^H(l)\boldsymbol{P}^{-1}\big]\boldsymbol{y}_p(l)\boldsymbol{y}_p^H(l) + (1-\gamma)\sum_{l=1}^{n}\Big(\mathrm{tr}\big[\boldsymbol{x}_z(l)\boldsymbol{x}_z^H(l)\boldsymbol{P}^{-1}\big] + \alpha_l^2\Big)\boldsymbol{y}_z(l)\boldsymbol{y}_z^H(l)$$

$$\boldsymbol{N}_\gamma(\boldsymbol{P}) = \gamma \sum_{l=1}^{n} \mathrm{tr}\big[\boldsymbol{y}_p(l)\boldsymbol{y}_p^H(l)\boldsymbol{P}\big]\boldsymbol{x}_p(l)\boldsymbol{x}_p^H(l) + (1-\gamma)\sum_{l=1}^{n}\Big(\mathrm{tr}\big[\boldsymbol{y}_z(l)\boldsymbol{y}_z^H(l)\boldsymbol{P}\big] + \beta_l^2\Big)\boldsymbol{x}_z(l)\boldsymbol{x}_z^H(l).$$

Setting $\partial I_\gamma(\boldsymbol{P},\xi)/\partial \boldsymbol{P} = \boldsymbol{0}$ and $\partial I_\gamma(\boldsymbol{P},\xi)/\partial\xi = 0$, it follows that

$$\boldsymbol{P}\,\boldsymbol{M}_\gamma(\boldsymbol{P})\boldsymbol{P} = \boldsymbol{G}_\gamma(\boldsymbol{P},\xi), \qquad \mathrm{tr}[\boldsymbol{K}_c\boldsymbol{P}^{-1}] = n \tag{22}$$

where

$$\boldsymbol{G}_\gamma(\boldsymbol{P},\xi) = \boldsymbol{N}_\gamma(\boldsymbol{P}) + \xi\boldsymbol{K}_c.$$

In order to solve the highly nonlinear equations in (22), we propose an iterative technique which starts with an initial iterate $\boldsymbol{P}_0$ and relaxes the first equation in (22) into a recursive second-order matrix equation

$$\boldsymbol{P}_{k+1}\boldsymbol{M}_\gamma(\boldsymbol{P}_k)\boldsymbol{P}_{k+1} = \boldsymbol{G}_\gamma(\boldsymbol{P}_k,\xi_{k+1}) \tag{23}$$

where $\boldsymbol{P}_k$ is known from the previous recursion. Solving (23) for a symmetric and positive-definite $\boldsymbol{P}_{k+1}$, we obtain

$$\boldsymbol{P}_{k+1} = \boldsymbol{M}_\gamma(\boldsymbol{P}_k)^{-\frac{1}{2}}[\boldsymbol{M}_\gamma(\boldsymbol{P}_k)^{\frac{1}{2}}\boldsymbol{G}_\gamma(\boldsymbol{P}_k,\xi_{k+1})\boldsymbol{M}_\gamma(\boldsymbol{P}_k)^{\frac{1}{2}}]^{\frac{1}{2}} \cdot \boldsymbol{M}_\gamma(\boldsymbol{P}_k)^{-\frac{1}{2}}. \tag{24}$$

The Lagrange multiplier ξ_{k+1} can be efficiently obtained using a bisection method [8] so that

$$f(\gamma,\xi_{k+1}) = \Big|n - \mathrm{tr}\big[\tilde{\boldsymbol{K}}_k\tilde{\boldsymbol{G}}_k(\gamma,\xi_{k+1})\big]\Big| < \varepsilon \tag{25}$$

is met, where

$$\tilde{\boldsymbol{G}}_k(\gamma,\xi_{k+1}) = [\boldsymbol{M}_\gamma(\boldsymbol{P}_k)^{\frac{1}{2}}\boldsymbol{G}_\gamma(\boldsymbol{P}_k,\xi_{k+1})\boldsymbol{M}_\gamma(\boldsymbol{P}_k)^{\frac{1}{2}}]^{-\frac{1}{2}}$$

$$\tilde{\boldsymbol{K}}_k = \boldsymbol{M}_\gamma(\boldsymbol{P}_k)^{\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{M}_\gamma(\boldsymbol{P}_k)^{\frac{1}{2}}.$$

This iteration process continues until

$$\Big|I_\gamma\big(\boldsymbol{P}_{k+1},\xi_{k+1}\big) - I_\gamma\big(\boldsymbol{P}_k,\xi_k\big)\Big| < \varepsilon \tag{26}$$

is satisfied for a prescribed tolerance $\varepsilon > 0$. If the iteration is terminated at step k, we set $\boldsymbol{P} = \boldsymbol{P}_k$ and claim it to be a solution.

B. Derivation of $\boldsymbol{T}$ Satisfying l_2-Scaling Constraints from $\boldsymbol{P}$

Having obtained an optimal $\boldsymbol{P}$, we now turn our attention to the construction of the optimal coordinate transformation matrix $\boldsymbol{T}$ that solves the problem of minimizing (16) subject to l_2-scaling constraints in (19). As is analyzed earlier, the optimal $\boldsymbol{T}$ assumes the form

$$\boldsymbol{T} = \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{U} \tag{27}$$

where $\boldsymbol{U}$ is an $n\times n$ orthogonal matrix that can be determined to satisfy the l_2-scaling constraints as follows. From (18) and (27) it follows that

$$\overline{\boldsymbol{K}}_c = \boldsymbol{U}^T\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{U}. \tag{28}$$

In order to find an $n\times n$ orthogonal matrix $\boldsymbol{U}$ such that the matrix $\overline{\boldsymbol{K}}_c$ satisfies l_2-scaling constraints in (19), we perform eigenvalue-eigenvector decomposition for the symmetric positive-definite matrix $\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{P}^{-\frac{1}{2}}$ as

$$\boldsymbol{P}^{-\frac{1}{2}}\boldsymbol{K}_c\boldsymbol{P}^{-\frac{1}{2}} = \boldsymbol{R}\boldsymbol{\Theta}\boldsymbol{R}^T \tag{29}$$

where $\boldsymbol{\Theta} = \mathrm{diag}\{\theta_1,\theta_2,\cdots,\theta_n\}$ with $\theta_i > 0$ for all i and $\boldsymbol{R}$ is an $n\times n$ orthogonal matrix. Next, an $n\times n$ orthogonal matrix $\boldsymbol{S}$ such that

$$(\boldsymbol{S}\boldsymbol{\Theta}\boldsymbol{S}^T)_{ii} = 1 \quad \text{for} \quad i = 1,2,\cdots,n \tag{30}$$

can be obtained by a numerical procedure [9] (p. 278). Using (28)-(30), it can be readily verified that the orthogonal matrix $\boldsymbol{U} = \boldsymbol{R}\boldsymbol{S}^T$ leads to matrix $\overline{\boldsymbol{K}}_c$ in (28) whose diagonal elements are equal to unity, hence the l_2-scaling constraints in (19) are satisfied. This matrix $\boldsymbol{U}$ together with (27) yields the solution of the problem of minimizing (16) subject to the l_2-scaling constraints in (19) as

$$\boldsymbol{T} = \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{S}^T. \tag{31}$$

IV. A Numerical Example

Consider a 4th-order state-space digital filter $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c},d)_4$ in (1) studied in [4] (p. 151)

$$\boldsymbol{A}^T = \begin{bmatrix} 3.5906 & 1 & 0 & 0 \\ -4.8535 & 0 & 1 & 0 \\ 2.9260 & 0 & 0 & 1 \\ -0.6636 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{c} = 10^{-3}\begin{bmatrix} 0.2353 & 0.0355 & 0.2147 & 0.0104 \end{bmatrix}$$

$$d = 0.1425$$

for which the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{Z}$ were found to be

$$\lambda = 0.932750 \pm j0.134752, \quad 0.862550 \pm j0.056149$$

$$v = 1.044533 \pm j0.212868, \quad 0.749941 \pm j0.147041$$

respectively. By using (9c), we obtained

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = 10^{-3}\begin{bmatrix} 1.755627 \\ 1.755627 \\ 1.419538 \\ 1.419538 \end{bmatrix}, \quad \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = 10^{2}\begin{bmatrix} 2.597616 \\ 2.597616 \\ 2.477293 \\ 2.477293 \end{bmatrix}$$

which produce $J_z(\boldsymbol{P}) \geq \sum_{l=1}^{4}(1+|\alpha_l\beta_l|)^2 = 7.894107$.

The original pole and zero sensitivity measures in (9a) and (9b) were computed as

$$J_p = 1.723110\times 10^7, \qquad J_z = 3.352758\times 10^5.$$

The controllability Gramian was then computed using (17) as

$$\boldsymbol{K}_c = 10^5\begin{bmatrix} 2.114795 & 2.103883 & 2.071415 & 2.018180 \\ 2.103883 & 2.114795 & 2.103883 & 2.071415 \\ 2.071415 & 2.103883 & 2.114795 & 2.103883 \\ 2.018180 & 2.071415 & 2.103883 & 2.114795 \end{bmatrix}.$$

By setting $\gamma = 1$ in (16), $\varepsilon = 10^{-8}$ in (26), and choosing $\boldsymbol{P}_0 = \boldsymbol{I}_4$ in (24) as an initial estimate, it took the proposed iterative algorithm 22 iterations to converge to

$$\boldsymbol{P} = 10^6 \begin{bmatrix} 0.951907 & 1.0477269 & 1.1418710 & 1.233192 \\ 1.047727 & 1.1696843 & 1.2926950 & 1.415602 \\ 1.141871 & 1.2926950 & 1.4480233 & 1.606767 \\ 1.233192 & 1.4156022 & 1.6067674 & 1.805765 \end{bmatrix}$$

which from (31) yields

$$\boldsymbol{T} = 10^2 \begin{bmatrix} -0.979221 & -7.631922 & 5.107309 & 3.146580 \\ -2.016850 & -8.259448 & 5.190007 & 4.212612 \\ -3.122063 & -8.865739 & 5.126772 & 5.492716 \\ -4.275411 & -9.454280 & 4.882113 & 7.005637 \end{bmatrix}$$

This optimized $\boldsymbol{T}$ leads to a state-space digital filter with

$$J_p(\boldsymbol{P}) = 4.000000, \qquad J_z(\boldsymbol{P}) = 97.343533.$$

This shows that minimal pole sensitivity was achieved subject to l_2-scaling constraints. For the equivalent realization, the controllability Gramian was found from (18) to be

$$\overline{\boldsymbol{K}}_c = \begin{bmatrix} 1.000000 & 0.436902 & 0.756058 & 0.823575 \\ 0.436902 & 1.000000 & 0.756058 & 0.823575 \\ 0.756058 & 0.756058 & 1.000000 & 0.950812 \\ 0.823575 & 0.823575 & 0.950812 & 1.000000 \end{bmatrix}$$

where the l_2-scaling constraints in (19) are satisfied.

The algorithm's performance in terms of a profile $I_\gamma(\boldsymbol{P}, \xi)$ in the first 22 iterations is shown in Fig. 1 from which the algorithm's rapid convergence can be observed. The corresponding numerical values of Lagrange multiplier ξ_i were found to be

$$\begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_4 & \xi_5 & \xi_6 \\ \xi_7 & \xi_8 & \xi_9 \\ \xi_{10} & \xi_{11} & \xi_{12} \end{bmatrix} = \begin{bmatrix} 7.82198 \times 10^5 & 1.75000 & 0.48733 \\ 0.23869 & 0.13462 & 0.07990 \\ 0.04849 & 0.02977 & 0.01838 \\ 0.01139 & 0.00707 & 0.00439 \end{bmatrix}$$

$$\begin{bmatrix} \xi_{13} & \xi_{14} & \xi_{15} \\ \xi_{16} & \xi_{17} & \xi_{18} \\ \xi_{19} & \xi_{20} & \xi_{21} \\ \xi_{22} & & \end{bmatrix} = 10^{-3} \begin{bmatrix} 2.73282 & 1.70054 & 1.05848 \\ 0.65895 & 0.41027 & 0.25545 \\ 0.15906 & 0.09905 & 0.06168 \\ 0.03841 & & \end{bmatrix}$$

Detailed numerical results of the optimized pole and zero sensitivity measures subject to l_2-scaling constraints corresponding to various values of γ are summarized in Table I.

We conclude this section with a remark on the numerical results summarized in Table I. Concerning the case of $\gamma = 1$ ($\gamma = 0$), it follows from (16) that the use of $\gamma = 1$ ($\gamma = 0$) simply excludes $J_z(\boldsymbol{P})$ ($J_p(\boldsymbol{P})$) in the optimization procedure and, as a result, the zero (pole) sensitivity went wildly large as shown in Table I. For practical system implementations, therefore, the use of γ in the range $0 < \gamma < 1$ is recommended.

V. CONCLUSION

In this paper, an efficient iterative technique for minimizing a weighted pole and zero sensitivity measure subject to the l_2-scaling constraints has been developed by relaxing the constraints into a single constraint on matrix trace and solving the relaxed problem with an effective matrix iteration scheme.

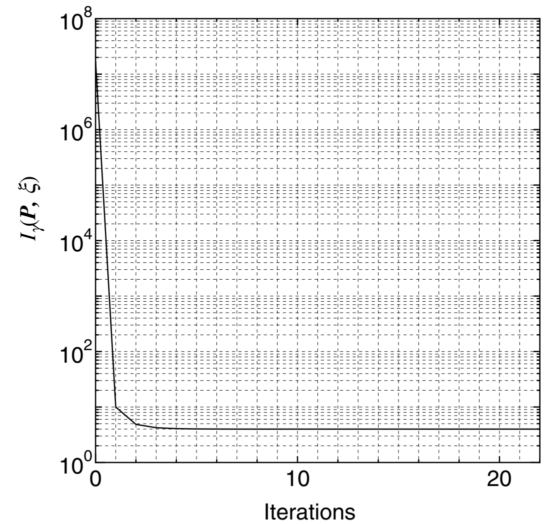

Fig. 1. Profile of $I_\gamma(\boldsymbol{P}, \xi)$ during the first 22 iterations with $\gamma = 1$

TABLE I
NUMERICAL RESULTS FOR VARIOUS WEIGHTING FACTOR

γ	$J_\gamma(\boldsymbol{P})$	$J_p(\boldsymbol{P})$	$J_z(\boldsymbol{P})$	$J_p(\boldsymbol{P})+J_z(\boldsymbol{P})$
1.0	4.000000	4.000000	97.343533	101.343533
0.9	8.146143	5.418422	32.695628	38.114050
0.8	10.371509	6.706021	25.033462	31.739483
0.7	11.925343	8.009571	21.062144	29.071716
0.6	13.019405	9.444172	18.382255	27.826426
0.5	13.725046	11.127489	16.322602	27.450091
0.4	14.055390	13.239312	14.599442	27.838754
0.3	13.977582	16.127815	13.056053	29.183868
0.2	13.390428	20.649124	11.575754	32.224878
0.1	12.012883	29.946083	10.020305	39.966388
0.0	8.224923	101.264589	8.224923	109.489512

Computer simulation results have demonstrated the validity and effectiveness of the proposed technique.

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