Roundoff Noise Analysis for Generalized Direct-Form II Structure of 2-D Separable-Denominator Digital Filters

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***Abstract*—Based on the concept of polynomial operators, generalized direct-form II structure of two-dimensional (2-D) separable-denominator digital filters is explored. It is shown that 2-D separable-denominator digital filters can be expressed by a generalized SIMO direct-form II and a generalized MISO transposed direct-form II connected in cascade. An expression for the roundoff noise gain in the resulting structure is then investigated. Moreover, the roundoff noise gain is compared with the one for the generalized direct-form II state-space realization of 2-D separable-denominator digital filters.**

I. Introduction

In the past decades, delta operator has been widely used in the realization of digital filters for the purpose of improvement of finite-word-length (FWL) performance in systems with high sampling rate [1]-[5]. Li and Gevers have studied the roundoff noise gain for the delta-operator state-space realization of a transfer function and under what conditions the roundoff noise gain for the optimal delta-operator realization is smaller than that for the optimal shift-operator realization [1]. In [2], the filter was expressed by second-order sections connected in cascade, each section was implemented with a direct form in delta operator, and different direct forms in the delta operator were extensively compared. In [3], the concept of separately scaling the Δs and filter coefficients in the delta transposed direct-form II section was proposed to globally further minimize output roundoff noise gain, as compared to selecting a single optimal Δ only. In [4], a method for obtaining an optimal arbitrary-order delta-operator direct-form II transposed filter has been presented in terms of the roundoff noise gain and coefficient sensitivity. Based on the concept of polynomial operators, a new structure which is considered a generalization of the shift operator z-based direct-form II transposed structure has been explored for digital filter implementation [5]. Alternatively, due to their desirable properties and ease of stability test, two-dimensional (2-D) separable-denominator digital filters have been widely studied as a very popular class for multidimensional signal processing in the past [6]-[12]. In addition, generalized direct-form II realization of 2-D separable-denominator digital filters has been constructed. Also, an expression for the roundoff noise gain, an l_2-scaling method to avoid overflow, and a way to minimize the roundoff noise gain with respect to free parameters subject to l_2-scaling constraints have been examined [13].

In this paper, the beginning part is written along the line of the paper in [13]. Unlike [13], the roundoff noise analysis is carried out for the generalized direct-form II structure of 2-D separable-denominator digital filters. First, motivated by [4]-[5] and based on the concept of polynomial operators, two sets of operators are introduced to express 2-D separable in denominator digital filters by a generalized SIMO direct-form II and a generalized MISO transposed direct-form II connected in cascade. An expression of the roundoff noise gain for the resulting structue is then investigated. Moreover, the roundoff noise gain is compare with the one for the generalized direct-form II state-space relization of 2-D separable-denominator digital filters in [13].

II. REALIZATION OF 2-D DIGITAL FILTERS

Consider a 2-D stable separable-denominator digital filter described by

$$H(z_1,z_2)=\frac{\displaystyle\sum_{k=0}^{m}\sum_{l=0}^{n}c_{kl}z_1^{-k}z_2^{-l}}{\displaystyle\Big(1+\sum_{k=1}^{m}a_kz_1^{-k}\Big)\Big(1+\sum_{l=1}^{n}b_lz_2^{-l}\Big)} \tag{1}$$

where the denominator and numerator are assumed to be co-prime. Let $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ be $(m+1)\times(m+1)$ and $(n+1)\times(n+1)$ nonsingular matrices, respectively, defined by

$$\begin{bmatrix} q_0^h(z_1)\\ q_1^h(z_1)\\ \vdots\\ q_m^h(z_1)\end{bmatrix}=\boldsymbol{P}_1\begin{bmatrix} z_1^m\\ \vdots\\ z_1\\ 1\end{bmatrix},\quad \begin{bmatrix} q_0^v(z_2)\\ q_1^v(z_2)\\ \vdots\\ q_n^v(z_2)\end{bmatrix}=\boldsymbol{P}_2\begin{bmatrix} z_2^n\\ \vdots\\ z_2\\ 1\end{bmatrix} \tag{2}$$

If scalars $\{\alpha_k|\, k=1,2,\cdots,m\}$, $\{\beta_l|\, l=1,2,\cdots,n\}$ and $\{r_{kl}|\, k=0,1,\cdots,m;\ l=0,1,\cdots,n\}$ are defined so that

$$\begin{aligned}
\kappa_1\,[1\ \alpha_1\ \alpha_2\ \cdots\ \alpha_m]\,\boldsymbol{P}_1&=[1\ a_1\ a_2\ \cdots\ a_m]\\
\kappa_2\,[1\ \beta_1\ \beta_2\ \cdots\ \beta_n]\,\boldsymbol{P}_2&=[1\ b_1\ b_2\ \cdots\ b_n]\\
\kappa_1\kappa_2\boldsymbol{P}_1^T\begin{bmatrix} r_{00}&\cdots&r_{0n}\\ \vdots&\ddots&\vdots\\ r_{m0}&\cdots&r_{mn}\end{bmatrix}\boldsymbol{P}_2&=\begin{bmatrix} c_{00}&\cdots&c_{0n}\\ \vdots&\ddots&\vdots\\ c_{m0}&\cdots&c_{mn}\end{bmatrix}
\end{aligned} \tag{3}$$

the transfer function in (1) can be written as

$$H(z_1,z_2)=\frac{\sum_{k=0}^{m}\sum_{l=0}^{n} r_{kl}q_k^h(z_1)q_l^v(z_2)}{\sum_{k=0}^{m}\alpha_k q_k^h(z_1)\sum_{l=0}^{n}\beta_l q_l^v(z_2)},\quad \alpha_0=\beta_0=1 \tag{4}$$

where κ_1 (κ_2) is a scaling factor such that $\alpha_0=1$ ($\beta_0=1$). Define

$$\begin{aligned} \rho_k^h(z_1) &= \frac{z_1-\overline{\gamma}_k}{\overline{\Delta}_k} \quad \text{for} \quad k=1,2,\cdots,m \\ \rho_l^v(z_2) &= \frac{z_2-\hat{\gamma}_l}{\hat{\Delta}_l} \quad \text{for} \quad l=1,2,\cdots,n \end{aligned} \tag{5}$$

where $\{\overline{\gamma}_k\}$, $\{\hat{\gamma}_l\}$, $\{\overline{\Delta}_k>0\}$ and $\{\hat{\Delta}_l>0\}$ are four sets of constants [13]. Let the polynomial operators be chosen as

$$\begin{aligned} q_k^h(z_1) &= \rho_k^h(z_1)\rho_{k+1}^h(z_1)\cdots\rho_m^h(z_1), \quad k=0,1,\cdots,m \\ q_l^v(z_2) &= \rho_l^v(z_2)\rho_{l+1}^v(z_2)\cdots\rho_n^v(z_2), \quad l=0,1,\cdots,n \\ q_m^h(z_1) &= q_n^v(z_2)=1, \qquad \rho_0^h(z_1)=\rho_0^v(z_2)=1 \end{aligned} \tag{6}$$

From (6), it follows that for $k,l\ge 1$,

$$\begin{aligned} \frac{q_k^h(z_1)}{q_0^h(z_1)} &= \rho_1^h(z_1)^{-1}\rho_2^h(z_1)^{-1}\cdots\rho_k^h(z_1)^{-1} \\ \frac{q_l^v(z_2)}{q_0^v(z_2)} &= \rho_1^v(z_2)^{-1}\rho_2^v(z_2)^{-1}\cdots\rho_l^v(z_2)^{-1} \end{aligned} \tag{7}$$

Hence the transfer function in (4) can be written as [13]

$$H(z_1,z_2)=\frac{\sum_{k=0}^{m}\sum_{l=0}^{n} r_{kl}\prod_{p=0}^{k}\rho_p^h(z_1)^{-1}\prod_{q=0}^{l}\rho_q^v(z_2)^{-1}}{\Big(\sum_{k=0}^{m}\alpha_k\prod_{p=0}^{k}\rho_p^h(z_1)^{-1}\Big)\Big(\sum_{l=0}^{n}\beta_l\prod_{q=0}^{l}\rho_q^v(z_2)^{-1}\Big)} \tag{8}$$

The implementations of $\rho_k^h(z_1)^{-1}$ and $\rho_l^v(z_2)^{-1}$ are depicted in Fig. 1. As an illustrative example, referring to shift-operator structure of a 2-D separable-denominator filter with $(m,n)=(3,2)$ shown in [14] (p. 956), the generalized filter structure in (8) with $(m,n)=(3,3)$ is illustrated in Fig. 2 where $u(i,j)$ is a scalar input and $y(i,j)$ is a scalar output.

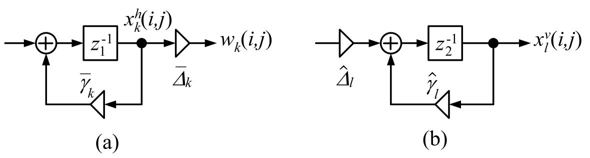

Fig. 1. (a) Implementation of $\rho_k^h(z_1)^{-1}$ and (b) that of $\rho_l^v(z_2)^{-1}$.

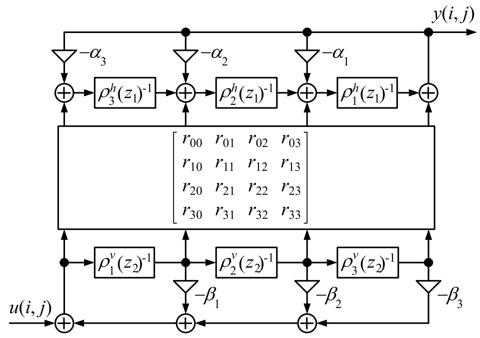

Fig. 2. The 2-D filter structure of (8) with $(m,n)=(3,3)$.

From Figures 1 and 2, we obtain

$$\begin{aligned} y(i,j) &= w_1(i,j)+r_{00}\Big[u(i,j)-\sum_{l=1}^{n}\beta_l x_l^v(i,j)\Big] \\ &\quad +\sum_{l=1}^{n} r_{0l}x_l^v(i,j) \\ w_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{w_{k+1}(i,j)+\sum_{l=1}^{n} r_{kl}x_l^v(i,j) \\ &\quad +r_{k0}\Big[u(i,j)-\sum_{l=1}^{n}\beta_l x_l^v(i,j)\Big]-\alpha_k y(i,j)\Big\} \end{aligned} \tag{9}$$

for $k=1,2,\cdots,m$ where $w_{m+1}(i,j)=0$. In addtion,

$$\begin{aligned} x_1^v(i,j+1) &= \hat{\gamma}_1 x_1^v(i,j)+\hat{\Delta}_1\Big[u(i,j)-\sum_{l=1}^{n}\beta_l x_l^v(i,j)\Big] \\ x_l^v(i,j+1) &= \hat{\Delta}_l x_{l-1}^v(i,j)+\hat{\gamma}_l x_l^v(i,j) \end{aligned} \tag{10}$$

for $l=2,3,\cdots,n$.

The structure in (8) has $2(m+n)+(m+1)(n+1)$ nontrivial parameters $\{\alpha_k\}$, $\{\overline{\Delta}_k\}$, $\{\beta_l\}$, $\{\hat{\Delta}_l\}$ and $\{r_{kl}\}$ plus $m+n$ free parameters $\{\overline{\gamma}_k\}$ and $\{\hat{\gamma}_l\}$.

III. ROUNDOFF NOISE ANALYSIS

We first investigate the effect of roundoff noise produced by the term $\alpha_k y(i,j)$, $1\le k\le m$ at the output. Due to the product quantization, for the actual filter implemented by a FWL machine, (9) can be written as

$$\begin{aligned} \tilde{y}(i,j) &= \tilde{w}_1(i,j)+r_{00}\Big[u(i,j)-\sum_{l=1}^{n}\beta_l x_l^v(i,j)\Big] \\ &\quad +\sum_{l=1}^{n} r_{0l}x_l^v(i,j) \\ \tilde{w}_s(i,j) &= \rho_s^h(z_1)^{-1}\Big\{\tilde{w}_{s+1}(i,j)+\sum_{l=1}^{n} r_{sl}x_l^v(i,j) \\ &\quad +r_{s0}\Big[u(i,j)-\sum_{l=1}^{n}\beta_l x_l^v(i,j)\Big]-\big[\alpha_s\tilde{y}(i,j)+\overline{\varepsilon}_s(i,j)\big]\Big\} \end{aligned} \tag{11}$$

for $1\le s\le m$ where $\tilde{w}_{m+1}(i,j)=0$, $\overline{\varepsilon}_s(i,j)=0$ unless $s=k$, $\tilde{y}(i,j)$ is the actual output, $\tilde{w}_s(i,j)$ is the actual signal of $w_s(i,j)$ and $\overline{\varepsilon}_k(i,j)=Q[\alpha_k\tilde{y}(i,j)]-\alpha_k\tilde{y}(i,j)$ is the roundoff noise due to quantizer $Q[\cdot]$. Subtracting (9) from (11) yields

$$\begin{aligned}\delta y(i,j) &= \delta w_1(i,j)\\ \delta w_s(i,j) &= \rho_s^h(z_1)^{-1}\left[\delta w_{s+1}(i,j) - \alpha_s \delta y(i,j) - \overline{\varepsilon}_s(i,j)\right]\end{aligned} \tag{12a}$$

where

$$\begin{aligned}\delta y(i,j) &= \tilde{y}(i,j) - y(i,j)\\ \delta w_s(i,j) &= \tilde{w}_s(i,j) - w_s(i,j) \\ &= \overline{\Delta}_s[\tilde{x}_s^h(i,j) - x_s^h(i,j)] = \overline{\Delta}_s\, \delta x_s^h(i,j)\end{aligned} \tag{12b}$$

If a 1-D state-space model $(\boldsymbol{A}_1, \overline{\boldsymbol{e}}_k, \boldsymbol{c}_1)_m$ is realized using (5) from (12a), the transfer function from $-\overline{\varepsilon}_k(i,j)$ to $\delta y(i,j)$ is given by

$$H_{1k}(z_1) = \boldsymbol{c}_1(z_1 \boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\overline{\boldsymbol{e}}_k \quad \text{for} \quad k = 1, 2, \cdots, m \tag{13}$$

where $\overline{\boldsymbol{e}}_k$ is the kth column of an identity matrix $\boldsymbol{I}_m$ and

$$\boldsymbol{c}_1 = \begin{bmatrix} \overline{\Delta}_1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} -\alpha_1\overline{\Delta}_1 & \overline{\Delta}_2 & \cdots & 0 \\ -\alpha_2\overline{\Delta}_1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \overline{\Delta}_m \\ -\alpha_m\overline{\Delta}_1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} \overline{\gamma}_1 & 0 & \cdots & 0 \\ 0 & \overline{\gamma}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \overline{\gamma}_m \end{bmatrix}$$

We define the normalized roundoff noise gain in terms of $H_{1k}(z_1)$ as

$$J_1(\alpha_k) = \frac{E[\delta y(i,j)^2]}{E[\overline{\varepsilon}_k(i,j)^2]} = \frac{1}{2\pi j}\oint_{|z_1|=1} H_{1k}^H(z_1)\, H_{1k}(z_1)\frac{dz_1}{z_1} \tag{14}$$

where $H_{1k}^H(z_1)$ denotes the conjugate transpose of $H_{1k}(z_1)$. Substituting (13) into (14), it follows that for $k = 1, 2, \cdots, m$

$$J_1(\alpha_k) = \overline{\boldsymbol{e}}_k^T \boldsymbol{W}^h \overline{\boldsymbol{e}}_k = \overline{\boldsymbol{e}}_k^T\left[\sum_{i=0}^{\infty}(\boldsymbol{c}_1\overline{\boldsymbol{A}}_1^i)^T\boldsymbol{c}_1\overline{\boldsymbol{A}}_1^i\right]\overline{\boldsymbol{e}}_k \tag{15}$$

where $\boldsymbol{W}^h$ is the horizontal observability Grammian which can be obtained by solving the Lyapunov equation [9],[13]

$$\boldsymbol{W}^h = \boldsymbol{A}_1^T \boldsymbol{W}^h \boldsymbol{A}_1 + \boldsymbol{c}_1^T\boldsymbol{c}_1$$

Similarly, the normalized roundoff noise gain produced by the coefficient r_{kl} for $l = 0, 1, \cdots, n$ in the second equation of (9) can be expressed as

$$J_2(r_{kl}) = \overline{\boldsymbol{e}}_k^T \boldsymbol{W}^h \overline{\boldsymbol{e}}_k \quad \text{for} \quad k = 1, 2, \cdots, m \tag{16}$$

With $w_{k+1}(i,j)$ replaced by $\overline{\Delta}_{k+1}x_{k+1}^h(i,j)$ in the second equation of (9), the normalized roundoff noise gain due to $\overline{\Delta}_{k+1}$ can be viewed as a function of r_{kl}, i.e.,

$$\begin{aligned} J_2(\overline{\Delta}_{k+1}) &= J_2(r_{kl}) = \overline{\boldsymbol{e}}_k^T \boldsymbol{W}^h \overline{\boldsymbol{e}}_k \\ &\text{for} \quad k = 1, 2, \cdots, m-1\end{aligned} \tag{17}$$

As shown in Fig. 1(a), parameter $\overline{\gamma}_k$ yields a multiplication $\overline{\gamma}_k x_k^h(i,j)$ which produces no roundoff noise if $\overline{\gamma}_k = 0, \pm 1$. Let $\psi(\overline{\gamma}_k)\epsilon_k^h(i,j)$ denote the roundoff noise due to $\overline{\gamma}_k$ where $\psi(\overline{\gamma}_k) = 1$ for all $\overline{\gamma}_k$ except $\overline{\gamma}_k = 0, \pm 1$ for which $\psi(\overline{\gamma}_k) = 0$, and let $\delta y(i,j)$ be the corresponding output deviation. Then the transfer function from $\psi(\overline{\gamma}_k)\epsilon_k^h(i,j)$ to $\delta y(i,j)$ becomes $H_{1k}(z_1)$ in (13). Actually, this roundoff noise can be viewed as the one generated by the term $r_{kl}x_l^v(i,j)$. Hence

$$J_3(\overline{\gamma}_k) = \psi(\overline{\gamma}_k) J_2(r_{kl}) = \psi(\overline{\gamma}_k)\overline{\boldsymbol{e}}_k^T \boldsymbol{W}^h \overline{\boldsymbol{e}}_k \tag{18}$$

for $k = 1, 2, \cdots, m$ where

$$\psi(\gamma) = \begin{cases} 1 & \text{for} \quad \gamma \neq 0\,, \pm 1 \\ 0 & \text{for} \quad \gamma = 0\,, \pm 1 \end{cases}$$

Regarding the coefficient r_{0l} for $l = 0, 1, \cdots, n$ in the first equation of (9), the first equation in (12a) is changed to

$$\delta y(i,j) = \delta w_1(i,j) + \varepsilon_{0l}(i,j) \tag{19}$$

When a 1-D state-space model $(\boldsymbol{A}_1, \boldsymbol{\alpha}, -\boldsymbol{c}_1, 1)_m$ is realized from (12a) whose first equation was replaced by (19) and such that $\overline{\varepsilon}_\xi(i,j) = 0$ in the second equation, the transfer function $H_{10}(z_1)$ from $\varepsilon_{0l}(i,j)$ to $\delta y(i,j)$ is given by

$$H_{10}(z_1) = -\boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{\alpha} + 1 \tag{20}$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \cdots, \alpha_m]^T$. Hence the normalized roundoff noise gain caused by the coefficient r_{0l} for $l = 0, 1, \cdots, n$ in the first equation of (9) is given by

$$J_4(r_{0l}) = \boldsymbol{\alpha}^T\boldsymbol{W}^h\boldsymbol{\alpha} + 1 \quad \text{for} \quad l = 0, 1, \cdots, n \tag{21}$$

Similarly, when $w_1(i,j)$ is replaced by $\overline{\Delta}_1 x_1^h(i,j)$ in the first equation of (9), the normalized roundoff noise gain due to $\overline{\Delta}_1$ can be considered as a function of r_{0l}, i.e.,

$$J_4(\overline{\Delta}_1) = J_4(r_{0l}) = \boldsymbol{\alpha}^T\boldsymbol{W}^h\boldsymbol{\alpha} + 1 \tag{22}$$

We now examine the effect of roundoff noise produced by the term $\beta_p x_p^v(i,j)$, $1 \leq p \leq n$ at the output. Due to the product quantization, for the actual filter implemented by a FWL machine, (9) and (10) can be written as

$$\begin{aligned}\tilde{y}(i,j) &= \tilde{w}_1(i,j) + \sum_{l=1}^{n} r_{0l}\tilde{x}_l^v(i,j) \\ &\quad + r_{00}\left[u(i,j) - \sum_{l=1}^{n}\beta_l\tilde{x}_l^v(i,j) - \hat{\varepsilon}_p(i,j)\right] \\ \tilde{w}_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{\tilde{w}_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl}\tilde{x}_l^v(i,j) \\ &\quad + r_{k0}\left[u(i,j) - \sum_{l=1}^{n}\beta_l\tilde{x}_l^v(i,j) - \hat{\varepsilon}_p(i,j)\right] - \alpha_k\tilde{y}(i,j)\Big\}\end{aligned} \tag{23}$$

for $k = 1, 2, \cdots, m$ where $\tilde{w}_{m+1}(i,j) = 0$, and

$$\begin{aligned}\tilde{x}_1^v(i,j+1) &= \hat{\gamma}_1\tilde{x}_1^v(i,j) \\ &\quad + \hat{\Delta}_1\left[u(i,j) - \sum_{l=1}^{n}\beta_l\tilde{x}_l^v(i,j) - \hat{\varepsilon}_p(i,j)\right] \\ \tilde{x}_l^v(i,j+1) &= \hat{\Delta}_l\tilde{x}_{l-1}^v(i,j) + \hat{\gamma}_l\tilde{x}_l^v(i,j)\end{aligned} \tag{24}$$

for $l = 2, 3, \cdots, n$, respectively, where $\tilde{x}_l^v(i,j)$ denotes the actual signal of $x_l^v(i,j)$ and $\hat{\varepsilon}_p(i,j) = Q[\beta_p\tilde{x}_p^v(i,j)] - \beta_p\tilde{x}_p^v(i,j)$ is the roundoff noise due to quantizer $Q[\cdot]$. Subtracting (9) from (23) yields

$$\begin{aligned}\delta y(i,j) &= \delta w_1(i,j) + \sum_{l=1}^{n} r_{0l}\delta x_l^v(i,j) \\ &\quad + r_{00}\left[-\hat{\varepsilon}_p(i,j) - \sum_{l=1}^{n}\beta_l\delta x_l^v(i,j)\right] \\ \delta w_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{\delta w_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl}\delta x_l^v(i,j) \\ &\quad + r_{k0}\left[-\hat{\varepsilon}_p(i,j) - \sum_{l=1}^{n}\beta_l\delta x_l^v(i,j)\right] - \alpha_k\delta y(i,j)\Big\}\end{aligned} \tag{25}$$

for $k = 1, 2, \cdots, m$ where $\delta w_{m+1}(i,j) = 0$ and $\delta x_l^v(i,j) = \tilde{x}_l^v(i,j) - x_l^v(i,j)$ for $l = 1, 2, \cdots, n$. Using (5) and (12b), we can write (25) as

$$\begin{aligned}
\delta y(i,j) &= \overline{\Delta}_1 \delta x_1^h(i,j) + \sum_{l=1}^{n} (r_{0l} - r_{00}\beta_l)\delta x_l^v(i,j) \\
&\quad + r_{00} \cdot -\hat{\varepsilon}_p(i,j) \\
\delta x_k^h(i+1,j) &= -\alpha_k \overline{\Delta}_1 \delta x_1^h(i,j) + \overline{\Delta}_{k+1}\delta x_{k+1}^h(i,j) \\
&\quad + \sum_{l=1}^{n} \big(r_{kl} - \alpha_k r_{0l} - r_{k0}\beta_l + r_{00}\alpha_k\beta_l\big)\delta x_l^v(i,j) \\
&\quad + \overline{\gamma}_k \delta x_k^h(i,j) + (r_{k0} - r_{00}\alpha_k) \cdot -\hat{\varepsilon}_p(i,j)
\end{aligned} \tag{26}$$

for $k = 1, 2, \cdots, m$ where $x_{m+1}^h(i,j) = 0$. By subtracting (10) from (24), we obtain

$$\begin{aligned}
\delta x_1^v(i,j+1) &= \hat{\gamma}_1 \delta x_1^v(i,j) + \hat{\Delta}_1 \Big[-\hat{\varepsilon}_p(i,j) - \sum_{l=1}^{n} \beta_l \delta x_l^v(i,j) \Big] \\
\delta x_l^v(i,j+1) &= \hat{\Delta}_l \delta x_{l-1}^v(i,j) + \hat{\gamma}_l \delta x_l^v(i,j)
\end{aligned} \tag{27}$$

When a 2-D local state-space model is realized from (26) and (27), the transfer function from $-\hat{\varepsilon}_p(i,j)$ to $\delta y(i,j)$ is found to be [13]

$$\begin{aligned}
H(z_1,z_2) &= d + \boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{b}_1 \\
&\quad + \big[\boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_2 + \boldsymbol{c}_2\big](z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\boldsymbol{b}_2
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\boldsymbol{A}_2 &= \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} - \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \begin{bmatrix} r_{01} & r_{02} & \cdots & r_{0n} \end{bmatrix} \\
&\quad - \begin{bmatrix} r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \end{bmatrix} [\beta_1 \ \beta_2 \cdots \beta_n] + r_{00} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} [\beta_1 \ \beta_2 \cdots \beta_n] \\
\boldsymbol{A}_4 &= \begin{bmatrix} -\beta_1\hat{\Delta}_1 & -\beta_2\hat{\Delta}_1 & \cdots & -\beta_n\hat{\Delta}_1 \\ \hat{\Delta}_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \hat{\Delta}_n & 0 \end{bmatrix} + \begin{bmatrix} \hat{\gamma}_1 & 0 & \cdots & 0 \\ 0 & \hat{\gamma}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \hat{\gamma}_n \end{bmatrix} \\
\boldsymbol{b}_1 &= \begin{bmatrix} r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \end{bmatrix} - r_{00} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} \hat{\Delta}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
\boldsymbol{c}_2 &= [\, r_{01} \ \ r_{02} \ \cdots \ r_{0n} \,] - r_{00} [\, \beta_1 \ \beta_2 \cdots \beta_n \,], \qquad d = r_{00}
\end{aligned}$$

We define the normalized noise gain in terms of $H(z_1,z_2)$ as

$$\begin{aligned}
J_5(\beta_p) &= \frac{E[\delta y(i,j)^2]}{E[\hat{\varepsilon}_p(i,j)^2]} \\
&= \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} H^H(z_1,z_2)\, H(z_1,z_2) \frac{dz_1 dz_2}{z_1 z_2}
\end{aligned} \tag{29}$$

Substituting (28) into (29) yields

$$\begin{aligned}
J_5(\beta_p) &= \boldsymbol{b}_1^T \boldsymbol{W}^h \boldsymbol{b}_1 + \boldsymbol{b}_2^T \boldsymbol{W}^v \boldsymbol{b}_2 + d^2 \\
&= \boldsymbol{b}_1^T \boldsymbol{W}^h \boldsymbol{b}_1 + \hat{\Delta}_1^2 \hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 + r_{00}^2
\end{aligned} \tag{30}$$

where $\boldsymbol{W}^v$ is the vertical observability Grammian which can be obtained by solving the Lyapunov equation [9], [13]

$$\boldsymbol{W}^v = \boldsymbol{A}_4^T \boldsymbol{W}^v \boldsymbol{A}_4 + \boldsymbol{A}_2^T \boldsymbol{W}^h \boldsymbol{A}_2 + \boldsymbol{c}_2^T \boldsymbol{c}_2$$

Due to the product quantization caused by $\hat{\Delta}_1$ in the first equation of (10), the actual filter implemented by a FWL machine can be written as

$$\begin{aligned}
\tilde{y}(i,j) &= \tilde{w}_1(i,j) + \sum_{l=1}^{n} r_{0l}\tilde{x}_l^v(i,j) \\
&\quad + r_{00}\Big[u(i,j) - \sum_{l=1}^{n} \beta_l \tilde{x}_l^v(i,j)\Big] \\
\tilde{w}_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{\tilde{w}_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl}\tilde{x}_l^v(i,j) \\
&\quad + r_{k0}\Big[u(i,j) - \sum_{l=1}^{n} \beta_l \tilde{x}_l^v(i,j)\Big] - \alpha_k \tilde{y}(i,j)\Big\}
\end{aligned} \tag{31}$$

for $k = 1, 2, \cdots, m$ where $\tilde{w}_{m+1}(i,j) = 0$, and

$$\begin{aligned}
\tilde{x}_1^v(i,j+1) &= \hat{\gamma}_1 \tilde{x}_1^v(i,j) + \hat{\Delta}_1 \Big[u(i,j) - \sum_{l=1}^{n} \beta_l \tilde{x}_l^v(i,j)\Big] \\
&\quad + \varepsilon_1^v(i,j) \\
\tilde{x}_l^v(i,j+1) &= \hat{\Delta}_l \tilde{x}_{l-1}^v(i,j) + \hat{\gamma}_l \tilde{x}_l^v(i,j)
\end{aligned} \tag{32}$$

for $l = 2, 3, \cdots, n$, respectively, where $\varepsilon_1^v(i,j)$ is the quantization error caused by $\hat{\Delta}_1$. Subtracting (9) from (31) yields

$$\begin{aligned}
\delta y(i,j) &= \delta w_1(i,j) + \sum_{l=1}^{n} r_{0l}\delta x_l^v(i,j) - r_{00}\sum_{l=1}^{n} \beta_l \delta x_l^v(i,j) \\
\delta w_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{\delta w_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl}\delta x_l^v(i,j) \\
&\quad - r_{k0}\sum_{l=1}^{n} \beta_l \delta x_l^v(i,j) - \alpha_k \delta y(i,j)\Big\}
\end{aligned} \tag{33}$$

for $k = 1, 2, \cdots, m$ where $\delta w_{m+1}(i,j) = 0$. Subtracting (10) from (32), we obtain

$$\begin{aligned}
\delta x_1^v(i,j+1) &= \hat{\gamma}_1 \delta x_1^v(i,j) - \hat{\Delta}_1 \sum_{l=1}^{n} \beta_l \delta x_l^v(i,j) + \varepsilon_1^v(i,j) \\
\delta x_l^v(i,j+1) &= \hat{\Delta}_l \delta x_{l-1}^v(i,j) + \hat{\gamma}_l \delta x_l^v(i,j)
\end{aligned} \tag{34}$$

for $l = 2, 3, \cdots, n$, respectively. Using (5) and (12b), it follows from (33) and (34) that the transfer function from $\varepsilon_1^v(i,j)$ to $\delta y(i,j)$ is found to be

$$H_{21}(z_1,z_2) = \big[\boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_2 + \boldsymbol{c}_2\big](z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\hat{\boldsymbol{e}}_1 \tag{35}$$

where $\hat{\boldsymbol{e}}_l$ denotes the lth column of an identity matrix $\boldsymbol{I}_n$. Hence the roundoff noise gain due to $\hat{\Delta}_1$ becomes

$$J_6(\hat{\Delta}_1) = \hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 \tag{36}$$

Similarly, the roundoff noise gain due to $\hat{\Delta}_l$ for $l = 2, 3, \cdots, n$ is given by

$$J_6(\hat{\Delta}_l) = \hat{\boldsymbol{e}}_l^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_l \quad \text{for} \quad l = 2, 3, \cdots, n \tag{37}$$

and the roundoff noise gain due to $\hat{\gamma}_l$ for $l = 1, 2, \cdots, n$ can be written as

$$J_6(\hat{\gamma}_l) = \psi(\hat{\gamma}_l)\hat{\boldsymbol{e}}_l^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_l \quad \text{for} \quad l = 1, 2, \cdots, n \tag{38}$$

Based on the above analysis, the total roundoff noise gain of the filter structure in Fig. 2 is defined as

$$\begin{aligned} J_\rho = &\sum_{k=1}^{m}\Big[J_1(\alpha_k)+\sum_{l=0}^{n}J_2(r_{kl})+J_3(\overline{\gamma}_k)\Big] \\ &+\sum_{k=1}^{m-1}J_2(\overline{\Delta}_{k+1})+J_4(\overline{\Delta}_1)+J_4(r_{00}) \\ &+\sum_{l=1}^{n}\Big[J_4(r_{0l})+J_5(\beta_l)+J_6(\hat{\Delta}_l)+J_6(\hat{\gamma}_l)\Big] \end{aligned} \tag{39}$$

which can be written as

$$\begin{aligned} J_\rho = &(n+3)\,\mathrm{tr}\big[\boldsymbol{W}^h\big]-\overline{\boldsymbol{e}}_m^T\boldsymbol{W}^h\overline{\boldsymbol{e}}_m+\mathrm{tr}\big[\overline{\boldsymbol{\Psi}}\boldsymbol{W}^h\big] \\ &+(n+2)\big(\boldsymbol{\alpha}^T\boldsymbol{W}^h\boldsymbol{\alpha}+1\big)+\mathrm{tr}\big[\boldsymbol{W}^v\big]+\mathrm{tr}\big[\hat{\boldsymbol{\Psi}}\boldsymbol{W}^v\big] \\ &+n\big(\boldsymbol{b}_1^T\boldsymbol{W}^h\boldsymbol{b}_1+\hat{\Delta}_1^2\hat{\boldsymbol{e}}_1^T\boldsymbol{W}^v\hat{\boldsymbol{e}}_1+r_{00}^2\big) \end{aligned} \tag{40}$$

where

$$\begin{aligned} \overline{\boldsymbol{\Psi}} &= \mathrm{diag}\{\psi(\overline{\gamma}_1),\psi(\overline{\gamma}_2),\cdots,\psi(\overline{\gamma}_m)\} \\ \hat{\boldsymbol{\Psi}} &= \mathrm{diag}\{\psi(\hat{\gamma}_1),\psi(\hat{\gamma}_2),\cdots,\psi(\hat{\gamma}_n)\} \end{aligned}$$

Remark 1: The roundoff noise gain for the state-space realization of the filter in (8) can be evaluated by [13]

$$\begin{aligned} J_{S\rho} = &(n+3)\mathrm{tr}[\boldsymbol{W}^h]-\overline{\boldsymbol{e}}_m^T\boldsymbol{W}^h\overline{\boldsymbol{e}}_m \\ &+\sum_{k=2}^{m}\psi(\overline{\gamma}_k)\overline{\boldsymbol{e}}_k^T\boldsymbol{W}^h\overline{\boldsymbol{e}}_k+n\hat{\boldsymbol{e}}_1^T\boldsymbol{W}^v\hat{\boldsymbol{e}}_1 \\ &+\mathrm{tr}[\boldsymbol{W}^v]+\sum_{l=2}^{n}\psi(\hat{\gamma}_l)\hat{\boldsymbol{e}}_l^T\boldsymbol{W}^v\hat{\boldsymbol{e}}_l+n+2 \end{aligned} \tag{41}$$

From (40) and (41), it follows that

$$\begin{aligned} J_\rho - J_{S\rho} = &\psi(\overline{\gamma}_1)\overline{\boldsymbol{e}}_1^T\boldsymbol{W}^h\overline{\boldsymbol{e}}_1+\psi(\hat{\gamma}_1)\hat{\boldsymbol{e}}_1^T\boldsymbol{W}^v\hat{\boldsymbol{e}}_1 \\ &+(n+2)\boldsymbol{\alpha}^T\boldsymbol{W}^h\boldsymbol{\alpha} \\ &+n\Big[\boldsymbol{b}_1^T\boldsymbol{W}^h\boldsymbol{b}_1+(\hat{\Delta}_1^2-1)\hat{\boldsymbol{e}}_1^T\boldsymbol{W}^v\hat{\boldsymbol{e}}_1+r_{00}^2\Big] \end{aligned} \tag{42}$$

IV. Conclusion

Based on the concept of polynomial operators, generalized direct-form II structure for 2-D separable-denominator digital filters has been explored by expressing a 2-D separable-denominator digital filter in terms of a generalized SIMO direct-form II and a generalized MISO transposed direct-form II connected in cascade. An expression of the roundoff noise gain for the resulting structure has been investigated. Moreover, the roundoff noise gain has been compared with the one for the generalized direct-form II state-space realization of 2-D separable-denominator digital filters. With $m+n$ free parameters $\{\overline{\gamma}_k\}$ and $\{\hat{\gamma}_l\}$, we can enjoy more degrees of freedom to minimize the FWL effects [13]. Numerical data has been excuded due to the limited space of the paper.

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