Robust Digital Filters Part 1 - Minimax FIR Filters

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***Abstract*—The paper is the first of a series of research investigations on robust digital filters which refer to filters that offer optimal performance under variations of filter parameters. We begin with quantitative characterization of performance robustness of a digital filter against parameter uncertainties. This is followed by several properties of the proposed robust performance measures and design formulations of robust FIR filters in L_2 (least-squares) and L_∞ (minimax) sense as nonsmooth convex problems. We present an accelerated subgradient algorithm for the design of L_∞-robust FIR filters with technical details involved in implementing the proposed algorithm. A numerical example is included for illustration of the proposed design method and performance evaluation in comparison with conventional minimax FIR filters.**

I. Introduction

Digital filters are at the core of many systems that collect, store, and process digital data. Analysis and design of digital filters that meet various requirements for the system to function properly has been a subject of study for several decades [1]. In a typical procedure, a filter is designed under the assumption of infinite precision and this is followed by an implementation phase to deal with finite-word-length (FWL) effects which include for example roundoff noise and power-of-two (Po2) constraints imposed upon filters parameters. There are also work where the filters with Po2 coefficients are investigated in the design phase [2], [3]. The design however either involves complicated mixed-integer linear programming [2] or does not guarantee optimality [3]. In this paper, we are concerned with digital filters with performance guarantee even if their parameters vary independently to a certain extent. Evidently, studying such filters allows us to combine the design and finite-precision implementation phases into a single procedure that covers a variety of scenarios including (but not limited to) Po2 constraints and FWL issues in a unified setting in terms of parameter uncertainties. In what follows we refer the class of such filters to *robust filters*. This paper is the first of a series of investigations of robust digital filters, which focuses on the design of robust minimax linear-phase FIR filters.

In Sec. II, motivation of our study is presented in a quantitative manner, and measures of robust performance are defined and their properties are examined. An algorithm for robust minimax FIR filters based on an accelerated subgradient method is presented in Sec. III. A detailed account of how subgradient of the objective function is evaluated is also provided there. Sec. IV presents an example to illustrate the design algorithm and evaluate its performance in comparison with conventional minimax design. Concluding remarks are given in Sec. V.

II. Robust Digital Filters

A. Motivation

Let $H(z)$ be the transfer function of a digital filter and $\boldsymbol{x}$ be a vector that collects the coefficients of $H(z)$. Throughout we denote the frequency response of the filter by $H(\omega, \boldsymbol{x})$. Given a desired frequency response, denoted by $H_d(\omega)$, the performance of a digital filter is typically evaluated in the frequency domain by an error measure that quantifies the closeness between $H(\omega, \boldsymbol{x})$ and $H_d(\omega)$ in a function space. Standard least-squares and minimax designs, for example, correspond to the choices of function space L_2 and L_∞, respectively, where the error measures are given by

$$\left[\int_\Omega W(\omega)|H(\omega, \boldsymbol{x}) - H_d(\omega)|^2 d\omega\right]^{1/2} \tag{1a}$$

and

$$\max_{\omega\in\Omega} W(\omega)|H(\omega, \boldsymbol{x}) - H_d(\omega)| \tag{1b}$$

where Ω denotes a frequency region of interest and $W(\omega) \geq 0$ is a weight function defined over Ω. For illustration purposes, let $\boldsymbol{x}^*$ be a minimizer of the L_∞ measure in (1b). The optimal performance of the filter, namely $H(\omega, \boldsymbol{x}^*)$, would be achieved only if its implementation is perfectly accurate. In practice, however, neither hardware nor software utilized in a practical implementation are of infinite precision, thus only an *approximate* version of $H(\omega, \boldsymbol{x}^*)$ is actually realized. This approximation may be modeled as a frequency response $H(\omega, \boldsymbol{x}^* + \boldsymbol{\delta})$ for some variation $\boldsymbol{\delta}$ due to various reasons ranging from power-of-two constraints on filter coefficients to rounding errors in multiplications using fixed-point arithmetic. Since the perturbed design represented by $\boldsymbol{x}^* + \boldsymbol{\delta}$ is no longer a minimizer of the error measure in (1b), the performance degradation at $\boldsymbol{x}^* + \boldsymbol{\delta}$ is inevitable even for small $\boldsymbol{\delta}$.

B. Measures for Performance Robustness

Motivated by the analysis above, we seek to develop design methods for digital filters that achieve performance optimality subject to variations of filter coefficients. The development begins with introducing new L_2 and L_∞ error measures that well fit into design formulations for filters with robust

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performance against coefficient variations. To this end, we define

$$e_2(\boldsymbol{x}) = \max_{\boldsymbol{\delta}\in B_r}\left[\int_\Omega W(\omega)|H(\omega,\boldsymbol{x}+\boldsymbol{\delta}) - H_d(\omega)|^2 d\omega\right]^{1/2} \quad (2a)$$

and

$$e_\infty(\boldsymbol{x}) = \max_{\boldsymbol{\delta}\in B_r}\max_{\boldsymbol{\omega}\in\Omega} W(\omega)|H(\omega,\boldsymbol{x}+\boldsymbol{\delta}) - H_d(\omega)| \quad (2b)$$

where B_r denotes a small bounding box or ball so that $\boldsymbol{x}+\boldsymbol{\delta}$ with $\boldsymbol{\delta}\in B_r$ defines a small region centered at given design $\boldsymbol{x}$ in parameter space. Evidently, functions $e_2(\boldsymbol{x})$ in (2a) and $e_\infty(\boldsymbol{x})$ in (2b) are respectively the largest L_2 and L_∞ errors over the region $\{\boldsymbol{x}+\boldsymbol{\delta}$ with $\boldsymbol{\delta}\in B_r\}$. Let $\boldsymbol{x}$ be a vector set B_r of dimension $K+1$ and denote $\boldsymbol{\delta} = [\delta_0\ \delta_1\ \cdots\ \delta_K]^T$. Typical choices of compact set B_r in (2) include bounding box

$$B_r = \{\boldsymbol{\delta} : |\delta_i| \le r_i, i = 0,\ 1,\ \ldots,\ K\} \quad (3a)$$

where $r_i \ge 0$ are constants, ball

$$B_r = \{\boldsymbol{\delta} : \|\boldsymbol{\delta}\|_2 \le r\} \quad (3b)$$

where $r \ge 0$ is a constant, and ellipsoid

$$B_r = \{\boldsymbol{\delta} : \|\boldsymbol{D}\boldsymbol{\delta}\|_2 \le 1\} \quad (3c)$$

where $\boldsymbol{D} = \text{diag}\{d_0, d_1, \ldots, d_K\}$ with $d_i \ge 0$.

C. Design Formulations

Given filter's order and type (FIR or IIR) and permitted parameter variation in terms of B_r in (3), robust lease-square and minimax designs are obtained by minimizing the error measure for performance robustness, namely by solving the problems

$$\text{minimize}\ \ e_2(\boldsymbol{x}) \quad (4a)$$

and

$$\text{minimize}\ \ e_\infty(\boldsymbol{x}) \quad (4b)$$

respectively, where $e_2(\boldsymbol{x})$ and $e_\infty(\boldsymbol{x})$ are defined in (2a) and (2b). Note that constraints on stability of $H(z)$ need to be imposed for IIR designs.

As the first part of our study on robust filters, this paper addresses the class of linear phase FIR filters only. For clarity, throughout we consider FIR filters of odd length N whose transfer function and frequency response are given by

$$H(z) = \sum_{i=0}^{N-1} h_i z^{-i} \text{ and } H(\omega,\boldsymbol{x}) = e^{-jK\omega}\boldsymbol{c}(\omega)^T\boldsymbol{x} \quad (5a)$$

where $K = (N-1)/2$ denotes filter's group delay,

$$\boldsymbol{c}(\omega) = \begin{bmatrix} 1 \\ \cos\omega \\ \vdots \\ \cos K\omega \end{bmatrix} \text{ and } \boldsymbol{x} = \begin{bmatrix} h_K \\ 2h_{K-1} \\ \vdots \\ 2h_0 \end{bmatrix} \quad (5b)$$

D. Properties of $e_2(\boldsymbol{x})$ and $e_\infty(\boldsymbol{x})$

For linear phase FIR filters, the desired frequency response typically assumes the form $H_d(\omega) = e^{-jK\omega}A_d(\omega)$ with $A_d(\omega)$ a desired amplitude response. From (2) and (5), it follows that

$$e_2(\boldsymbol{x}) = \max_{\boldsymbol{\delta}\in B_r}\left[\int_\Omega W(\omega)|\boldsymbol{c}(\omega)^T(\boldsymbol{x}+\boldsymbol{\delta}) - A_d(\omega)|^2 d\omega\right]^{1/2} \quad (6a)$$

and

$$e_\infty(\boldsymbol{x}) = \max_{\boldsymbol{\delta}\in B_r}\max_{\boldsymbol{\omega}\in\Omega} W(\omega)|\boldsymbol{c}(\omega)^T(\boldsymbol{x}+\boldsymbol{\delta}) - A_d(\omega)| \quad (6b)$$

Property 1 Functions $e_2(\boldsymbol{x})$ and $e_\infty(\boldsymbol{x})$ in (6) are convex

The property follows from definition of a function being convex: for any points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and a scalar $\lambda\in[0,\ 1]$, we have

$$\begin{aligned}
&e_2(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \\
&= \max_{\boldsymbol{\delta}\in B_r}\left[\int_\Omega W(\omega)|\boldsymbol{c}(\omega)^T(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 + \boldsymbol{\delta}) \right. \\
&\qquad\qquad \left. -A_d(\omega)|^2 d\omega\right]^{1/2} \\
&= \max_{\boldsymbol{\delta}\in B_r}\Big\|\lambda W(\omega)^{1/2}\left[\boldsymbol{c}(\omega)^T(\boldsymbol{x}_1+\boldsymbol{\delta}) - A_d(\omega)\right] \\
&\quad +(1-\lambda)W(\omega)^{1/2}\left[\boldsymbol{c}(\omega)^T(\boldsymbol{x}_2+\boldsymbol{\delta}) - A_d(\omega)\right]\Big\|_2 \\
&\le \lambda\max_{\boldsymbol{\delta}\in B_r}\Big\|W(\omega)^{1/2}\left[\boldsymbol{c}(\omega)^T(\boldsymbol{x}_1+\boldsymbol{\delta}) - A_d(\omega)\right]\Big\|_2 \\
&\quad +(1-\lambda)\max_{\boldsymbol{\delta}\in B_r}\Big\|W(\omega)^{1/2}\left[\boldsymbol{c}(\omega)^T(\boldsymbol{x}_2+\boldsymbol{\delta}) - A_d(\omega)\right]\Big\|_2 \\
&= \lambda e_2(\boldsymbol{x}_1) + (1-\lambda)e_2(\boldsymbol{x}_2)
\end{aligned}$$

and

$$\begin{aligned}
&e_\infty(\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \\
&= \max_{\boldsymbol{\delta}\in B_r}\max_{\omega\in\Omega}\big|\lambda W(\omega)\left[\boldsymbol{c}(\omega)^T(\boldsymbol{x}_1+\boldsymbol{\delta}) - A_d(\omega)\right] \\
&\quad +(1-\lambda)W(\omega)\left[\boldsymbol{c}(\omega)^T(\boldsymbol{x}_2+\boldsymbol{\delta}) - A_d(\omega)\right]\big| \\
&\le \lambda\max_{\boldsymbol{\delta}\in B_r}\max_{\omega\in\Omega} W(\omega)\left|\boldsymbol{c}(\omega)^T(\boldsymbol{x}_1+\boldsymbol{\delta}) - A_d(\omega)\right| \\
&\quad +(1-\lambda)\max_{\boldsymbol{\delta}\in B_r}\max_{\omega\in\Omega} W(\omega)\left|\boldsymbol{c}(\omega)^T(\boldsymbol{x}_2+\boldsymbol{\delta}) - A_d(\omega)\right| \\
&= \lambda e_\infty(\boldsymbol{x}_1) + (1-\lambda)e_\infty(\boldsymbol{x}_2)
\end{aligned}$$

Property 1 implies that both (4a) and (4b) are convex problems if the objective functions are given by (6a) and (6b). On the other hand, however, the functions in (6a) and (6b) are not differentiable because each is defined as *maximum* of a certain family of functions involving either *square roots* or operation of taking absolute values. Therefore, more precisely the problems of minimizing $e_2(\boldsymbol{x})$ and $e_\infty(\boldsymbol{x})$ in (6) are *nondifferentiable* convex problems. In the rest of the paper we focus on minimax design of robust FIR filters by solving problem (4b) where $e_\infty(\boldsymbol{x})$ is given by (6b) and B_r is given by (3a). The next property provides a formula for computing the *subdifferential* of $e_\infty(\boldsymbol{x})$ in (6b), which is a key term in the design method proposed in Sec. III.

Property 2 The subdifferential of $e_\infty(\boldsymbol{x})$ in (6b) with respect to $\boldsymbol{x}$ is given by

$$\partial e_\infty(\boldsymbol{x}) = W(\omega^*)\boldsymbol{c}(\omega^*)(\nabla|y|)_{y=\boldsymbol{c}(\omega^*)^T\boldsymbol{x}-[A_d(\omega^*)-\boldsymbol{c}(\omega^*)^T\boldsymbol{\delta}^*]} \quad (7a)$$

where ω^* and $\boldsymbol{\delta}^*$ are determined by

$$(\omega^*, \boldsymbol{\delta}^*) = \arg_{\omega,\boldsymbol{\delta}} \left(\max_{\boldsymbol{\delta}\in B_r}\max_{\omega\in\Omega} W(\omega)|\boldsymbol{c}(\omega)^T(\boldsymbol{x}+\boldsymbol{\delta}) - A_d(\omega)|\right) \quad (7b)$$

for a given $\boldsymbol{x}$, and

$$\nabla|y| = \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0 \\ \in [-1,\ 1] & \text{if } y = 0 \end{cases} \quad (7c)$$

To prove the property, note that the convexity of $W(\omega^*)|\boldsymbol{c}(\omega^*)^T(\boldsymbol{x}+\boldsymbol{\delta}^*) - A_d(\omega^*)|$ implies that [4]

$$W(\omega^*)|\boldsymbol{c}(\omega^*)^T(\tilde{\boldsymbol{x}}+\boldsymbol{\delta}^*) - A_d(\omega^*)| \geq e_\infty(\boldsymbol{x}) + (\tilde{\boldsymbol{x}}-\boldsymbol{x})^T\boldsymbol{g} \quad (8)$$

for any $\tilde{\boldsymbol{x}}$ and $\boldsymbol{x}$, where $\boldsymbol{g}$ is the subdifferential of $W(\omega^*)|\boldsymbol{c}(\omega^*)^T(\boldsymbol{x}+\boldsymbol{\delta}^*) - A_d(\omega^*)|$. By definition we also have

$$e_\infty(\tilde{\boldsymbol{x}}) \geq W(\omega^*)|\boldsymbol{c}(\omega^*)^T(\tilde{\boldsymbol{x}}+\boldsymbol{\delta}^*) - A_d(\omega^*)|$$

which in conjunction with (8) implies that $e_\infty(\tilde{\boldsymbol{x}}) \geq e_\infty(\boldsymbol{x}) + (\tilde{\boldsymbol{x}}-\boldsymbol{x})^T\boldsymbol{g}$, hence $\boldsymbol{g}$ is also the subdifferential of $e_\infty(\boldsymbol{x})$. It is known (see [4, p. 130]) that the subdifferential of $W(\omega^*)|\boldsymbol{c}(\omega^*)^T(\tilde{\boldsymbol{x}}+\boldsymbol{\delta}^*) - A_d(\omega^*)|$ is given by (7), which completes the proof. Technical details of determining ω^* and $\boldsymbol{\delta}^*$ in (7b) can be found in Sec. III.B.

III. AN ALGORITHM FOR ROBUST MINIMAX FIR FILTERS

A. The Algorithm

The design problem at hand is formulated in (4b) and (6b), which can be stated concisely as

$$\underset{x}{\text{minimize}} \max_{\boldsymbol{\delta}\in B_r}\max_{\omega\in\Omega} W(\omega)|\boldsymbol{c}(\omega)^T(\boldsymbol{x}+\boldsymbol{\delta}) - A_d(\omega)| \quad (9)$$

As shown in Sec. II, the objective function of (9) is convex but nonsmooth, as a result many algorithms that work well for smooth convex functions do not apply to (9). Among other things the subgradient method [4] suits (9) well because it only requires subgradient of the objective to proceed. In the kth iteration of the method, the filter's parameter vector $\boldsymbol{x}_k$ is updated to $\boldsymbol{x}_{k+1}$ as

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha_k\boldsymbol{g}_k$$

where $\boldsymbol{g}_k$ is a subgradient of $e_\infty(\boldsymbol{x})$ at $\boldsymbol{x}_k$, i.e. $\boldsymbol{g}_k \in \partial e_\infty(\boldsymbol{x}_k)$, and $\alpha_k > 0$ is a step size which will be addressed shortly. Like the gradient descent (GD) algorithm, the subgrandient method is known to be slow in practice. For the design problem at hand, we adopt a variant of the subgradient method, known as the heavy ball method ([5], p. 86) and GD with momentum, to improve convergence. The heavy ball algorithm updates iterate $\boldsymbol{x}_k$ to

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha_k\boldsymbol{g}_k + \beta_k(\boldsymbol{x}_k - \boldsymbol{x}_{k-1}) \quad (10)$$

where the iteration starts with $k = 0$ where point $\boldsymbol{x}_{-1}$ is set to be equal to $\boldsymbol{x}_0$ so the last term in (10) vanishes. The last term of (10) is often interpreted as one using past iterates to provide momentum that pushes the current iterate like a heavy ball to move down hill faster. Step size α_k in (10) is calculated using

$$\alpha_k = \frac{e_\infty(\boldsymbol{x}_k) - e_{best}^{(k)} + \gamma_k}{\|\boldsymbol{g}_k\|_2^2} \quad (11)$$

which was initiated by Polyak [6] where $e_{best}^{(k)} = \min_{i=1,\ldots,k} e_\infty(\boldsymbol{x}_i)$ keeps track of the best performance achieved so far, $\gamma_k > 0$ is a sequence satisfying $\sum_{k=0}^\infty \gamma_k = \infty$ and $\sum_{k=0}^\infty \gamma_k^2 < \infty$, and β_k satisfies $\sum_{k=0}^\infty \beta_k = \infty$ and $\sum_{k=0}^\infty \beta_k^2 < \infty$. In our design problems γ_k and β_k are simply set to be proportional to $1/(k+1)$.

From (11) we see that the major step of the algorithm is to calculate subgradient $\boldsymbol{g}_k$ using (7) where (7a) and (7c) are straightforward to implement, and below we address the nontrivial problem (7b).

B. Computing Subgradient $\boldsymbol{g}_k$

Let $\omega \in \Omega$ be an arbitrary and temporally fixed frequency. The function in (7b) can be expressed as

$$W(\omega)\left|\sum_{i=0}^K \cos(i\omega)\delta_i - \big(A_d(\omega) - \boldsymbol{c}(\omega)^T\boldsymbol{x}\big)\right| \quad (12)$$

Note that in (12) only δ_i's are variables and these variables appear separately and are bounded by $|\delta_i| \leq r_i$. Under the circumstances, it is clear that the maximum of the function in (12) with respect to $\boldsymbol{\delta} \in B_r$ given by (3a) is achieved when each δ_i is chosen according to the following rules: (i) if $A_d(\omega) - \boldsymbol{c}(\omega)^T\boldsymbol{x}$ and $\cos(i\omega)$ possess same signs, set $\delta_i = -r_i$; (ii) if $A_d(\omega)-\boldsymbol{c}(\omega)^T\boldsymbol{x}$ and $\cos(i\omega)$ possess opposite signs, set $\delta_i = r_i$. Based on this, the optimal $\boldsymbol{\delta}$ can be expressed as $\boldsymbol{\delta}(\omega) = -\text{sign}\big(A_d(\omega)-\boldsymbol{c}(\omega)^T\boldsymbol{x}\big)\{\text{sign}(\boldsymbol{c}(\omega))\}\cdot\boldsymbol{r}$ where $\boldsymbol{r} = [r_0\ r_1\ \cdots\ r_K]^T$ and "$\cdot$" denotes pointwise multiplication between two vectors. At $\boldsymbol{\delta}(\omega)$ the maximum value of (12) is equal to

$$W(\omega)\left(\sum_{i=0}^K r_i|\cos(i\omega)| + |A_d(\omega) - \boldsymbol{c}(\omega)^T\boldsymbol{x}|\right)$$

The problem of maximizing above function over frequency region Ω is nonconvex, however it is an one-dimensional problem which can be realistically solved by replacing Ω with a finite set Ω_d of frequency grids evenly placed over Ω, thus

$$\omega^* = \max_{\omega\in\Omega_d} W(\omega)\left(\sum_{i=0}^K r_i|\cos(i\omega)| + |A_d(\omega) - \boldsymbol{c}(\omega)^T\boldsymbol{x}|\right) \quad (13a)$$

and the optimal $\boldsymbol{\delta}$ is obtained as

$$\boldsymbol{\delta}^* = -\text{sign}\big(A_d(\omega^*) - \boldsymbol{c}(\omega^*)^T\boldsymbol{x}\big)\{\text{sign}(\boldsymbol{c}(\omega^*))\}\cdot\boldsymbol{r} \quad (13b)$$

We remark that quantities $W(\omega)$, $A_d(\omega)$, $\boldsymbol{c}(\omega)$, and $r_i|\cos(i\omega)|$ for $i = 0,\ 1,\ \ldots,\ K$ and $\omega \in \Omega_d$ that are required to compute ω^* in (13a) only need to compute once. The algorithm can now be summarized as follows.

Algorithm for Robust Minimax FIR Filters

inputs:

desired amplitude response $A_d(\omega)$, filter length N, weight $W(\omega)$, frequency grids Ω_d, initial design $\boldsymbol{x}_0$, and number of iterations N_t.

for $k = 0,\ 1,\ \ldots,\ N_t$,

  step 1: use (13) to compute ω^* and $\boldsymbol{\delta}^*$;

    use (7a) and (7c) to compute $\boldsymbol{g}_k$.

  step 2: use (10) and (11) to compute $\boldsymbol{x}_{k+1}$

end

IV. An Example

For illustration purposes consider designing a robust minimax lowpass FIR filter of length $N = 21$ with normalized passband edge $\omega_p = 0.4\pi$ and stopband edge $\omega_a = 0.5\pi$. For simplicity we assume $W(\omega) \equiv 1$ and a bounding box (3a) with $r_i = 0.005$ for $i = 0,\ 1,\ \ldots,\ 10$, namely

$$B_r = \{\boldsymbol{\delta} : |\delta_i| \leq 0.005,\ i = 0,\ 1,\ \ldots,\ 10\} \tag{14}$$

The set Ω_d consists of 100 frequency grids that are evenly placed over the union of the passband and stopband $[0,\ 0.4\pi] \bigcup [0.6\pi,\ \pi]$. We use a least-squares lowpass filter with the same passband and stopband edges as the initial point of the proposed algorithm. The parameters γ_k in (11) and β_k in (10) were set to

$$\gamma_k = \frac{0.16}{k+1} \quad \text{and} \quad \beta_k = \frac{0.08}{k+1}$$

The algorithm was able to reduce the object function from 0.2837 to 0.0906 in 100 iterations, see Fig. 1. However, it requires many more iterations to practically reach the global solution. Fig. 2 depicts the amplitude response of the robust filter obtained after 10^4 iterations, at which the objective function was reduced to $e_\infty(\boldsymbol{x}^*) = 0.0898$.

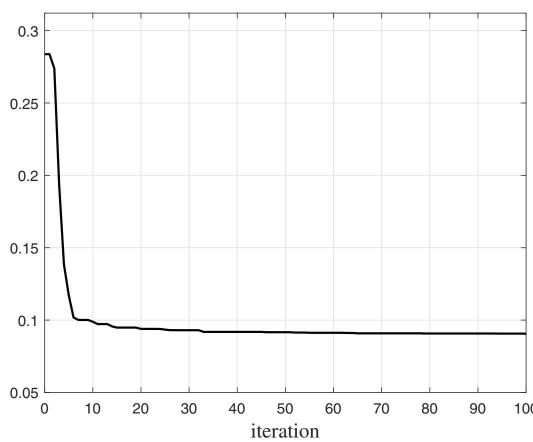

Fig. 1. Profile of objective function $e_\infty(\boldsymbol{x}_k)$ over the first 100 iterations.

For comparison, a conventional linear-phase minimax FIR filter of length 21 was designed using the Parks-McClellan (PM) algorithm [1]. Let $\boldsymbol{x}^{(PM)}$ be its parameter vector, using (7b), (13a), and (13b), the robustness measure of the PM filter defined by (6b) was found to be $e_\infty(\boldsymbol{x}^{(PM)}) = 0.1099$. This is to say, the approximation error of an FIR filter

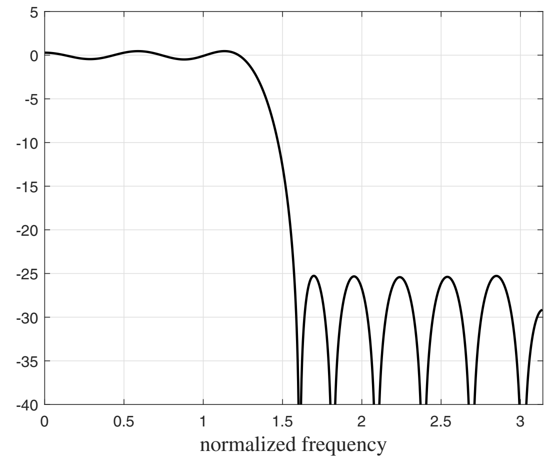

Fig. 2. Amplitude response of the robust minimax FIR filter.

whose coefficients vary from the PM filter $\boldsymbol{x}^{(PM)}$ within the bounding box B_r in (14) is bounded by 0.1099, while the approximation error of an FIR filter whose coefficients vary from the robust minimax filter $\boldsymbol{x}^*$ within the bounding box B_r in (14) is guaranteed not exceeding 0.0898, representing a 18.23% reduction. The first 11 coefficients of the robust minimax and PM filters are given in Table 1.

TABLE I First 11 Coefficients

Robust minimax	PM filter
0.037739257611475	0.038776212428159
0.003955891303322	0.002476747389280
−0.030947825075270	−0.030327895328723
−0.017617078016662	−0.018181020733260
0.034656023078736	0.035632966126753
0.040944804608671	0.039290923363811
−0.045087860127533	−0.045102726246706
−0.091116616564451	−0.092430917385436
0.046215874348786	0.047087467600674
0.313562923929549	0.311837559676779
0.449135853464320	0.448729827357856

V. Conclusions

We have proposed a framework to allow analysis and design of digital filters that offer guaranteed performance under parameter variations. Filters such as these are expected to be of use in practice as their performances are more robust subject to various constraints in their implementation. By focusing on the design of robust minimax FIR filters, we have shown how robust filters can be designed using nonsmooth convex optimization.

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