On $l_2$-Sensitivity for Generalized Direct-Form II Structure of 2-D Separable-Denominator Filters

Takao Hinamoto  
Hiroshima University  
Higashi-Hiroshima 739-8527, Japan  
Email: hinamoto@ieee.org

Akimitsu Doi  
Hiroshima Institute of Technology  
Hiroshima 731-5193, Japan  
Email: doi@cc.it-hiroshima.ac.jp

Wu-Sheng Lu  
University of Victoria  
Victoria, BC, Canada V8W 3P6  
Email: wslu@ece.uvic.ca

Abstract—A new expression of evaluating $l_2$-sensitivity for generalized direct-form II structure of two-dimensional (2-D) separable-denominator digital filters is derived and analyzed. Unlike the previous work, $l_2$-sensitivity is analyzed for the filter structure instead of its state-space realization. Then the resulting $l_2$-sensitivity measure is compared with that deduced in a recent study of generalized direct-form II state-space realization of 2-D separable-denominator digital filters. In a numerical example, the new $l_2$-sensitivity measure is minimized with respect to the free parameters subject to $l_2$-scaling constraints by an exhaustive search over a set of discrete values of finite cardinality, and the results are compared with those obtained by existing techniques.

I. INTRODUCTION

2-D separable-denominator digital filters are recognized as a very popular class of multi-dimensional signal processors due to the need in realizing a 2-D magnitude response with various symmetries of quarter-plane filters except for diagonal symmetry, easy stability test performed by simply calculating the poles of the two one-dimensional polynomials, and fewer multipliers required to realize the transfer function, especially in the case of symmetric magnitude response [1]-[4]. It is well known that the reduction of coefficient sensitivities is an important issue in the implementation of IIR digital filters using fixed-point arithmetic. For 2-D state-space digital filters whose transfer functions are separable in denominator, coefficient sensitivities have been analyzed using either a mixture of $l_1/l_2$ norms [5] or a pure $l_2$ norm [6],[7]. 2-D state-space filter structures with low coefficient sensitivities have also been synthesized [5]-[7]. Although the $l_2$-sensitivity minimization is technically more challenging, it is more natural and reasonable than the conventional $l_1/l_2$ mixed sensitivity minimization. Alternatively, delta-operator-based implementations have received considerable interest because of their excellent finite-word-length performance under fast sampling, and delta-operator-based designs have extensively studied in the area of digital signal processing and digital control systems[8]-[17]. Recently, generalized direct-form II structure and its state-space realization of 2-D separable-denominator digital filters have been presented [18]. In addition, the analysis and minimization of round-off noise have been explored [18],[19]. More recently, the problem of analyzing and minimizing the $l_2$-sensitivity for generalized direct-form II state-space realization of 2-D separable-denominator digital filters has been considered and investigated [20].

In this paper, we analyze the $l_2$-sensitivity for generalized direct-form II structure of 2-D separable-denominator digital filters. In [20], $l_2$-sensitivity analysis was performed not for the filter structure, but its state-space realization. The resulting $l_2$-sensitivity measure is then compared with that deduced in a recent study of generalized direct-form II state-space realization of 2-D separable-denominator digital filters [20]. In a numerical example, the new $l_2$-sensitivity measure is minimized with respect to the free parameters subject to $l_2$-scaling constraints by exhaustive search in a finite element space. The numerical results are compared with those obtained by the techniques in [7] and [20].

II. STRUCTURE OF 2-D DIGITAL FILTERS

Consider a stable 2-D separable-denominator digital filter described by

$$H(z_1, z_2) = \frac{\sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl} z_2^{-l} z_1^{-k}}{(z_2^{m} + \sum_{k=1}^{m} a_k z_1^{-k})(z_2^{n} + \sum_{l=1}^{n} b_l z_2^{-l})}$$  \hspace{1cm} (1)

where the denominator and numerator are assumed coprime. It was shown [18] that the transfer function in (1) can be converted into

$$H(z_1, z_2) = \frac{\sum_{k=0}^{m} \sum_{l=0}^{n} r_{kl} \prod_{p=0}^{k} \rho_{p}^{h}(z_1)^{-1} \prod_{q=0}^{l} \rho_{q}^{v}(z_2)^{-1}}{(\sum_{k=0}^{m} \sum_{p=0}^{k} \prod_{p=0}^{k} \rho_{p}^{h}(z_1)^{-1})(\sum_{l=0}^{n} \prod_{q=0}^{l} \rho_{q}^{v}(z_2)^{-1})}$$  \hspace{1cm} (2)

where $\alpha_0 = \beta_0 = 1$ and $\rho_{0}^{h}(z_1)^{-1} = \rho_{0}^{v}(z_2)^{-1} = 1$. The implementations of generalized delta operators $\rho_{k}^{h}(z_1)^{-1}$ and $\rho_{l}^{v}(z_2)^{-1}$ are depicted in Fig. 1. As an example, the generalized filter structure in (2) with $(m, n) = (3,3)$ is illustrated in Fig. 2 where $u(i, j)$ is a scalar input and $y(i, j)$ is a scalar output.

Fig. 1. (a) Implementation of $\rho_{0}^{h}(z_1)^{-1}$ and (b) that of $\rho_{0}^{v}(z_2)^{-1}$. 

978-1-7281-3320-1/20/$31.00 ©2020 IEEE
As is shown in [18], the 2-D filter in (2) can be realized by the Roesser model \( \{A_1, A_2, A_3, b_1, b_2, c_1, c_2, d\}\) \(m, n\) [21] as

\[
\begin{align*}
\begin{bmatrix}
    x^h(i+1, j) \\
x^v(i, j+1)
\end{bmatrix} &= 
\begin{bmatrix}
    A_1 & A_2 \\
    0 & A_4
\end{bmatrix}
\begin{bmatrix}
    x^h(i, j) \\
x^v(i, j)
\end{bmatrix} + 
\begin{bmatrix}
    b_1 \\
b_2
\end{bmatrix} u(i, j),
\end{align*}
\]

\[
y(i, j) = [c_1 \ c_2] \begin{bmatrix}
    x^h(i, j) \\
x^v(i, j)
\end{bmatrix} + d u(i, j)
\]

(3)

where

\[
\begin{align*}
x^h(i, j) &= \begin{bmatrix} x^h_1(i, j) \ x^h_2(i, j) \ \cdots \ x^h_m(i, j) \end{bmatrix}^T \\
x^v(i, j) &= \begin{bmatrix} x^v_1(i, j) \ x^v_2(i, j) \ \cdots \ x^v_m(i, j) \end{bmatrix}^T
\end{align*}
\]

\[
\begin{array}{c}
A_1 = \\
\begin{bmatrix}
    -\alpha_1 \Delta_1 & 0 & \cdots & 0 \\
    -\alpha_2 \Delta_1 & -\Delta_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    -\alpha_m \Delta_1 & 0 & \cdots & -\Delta_m
\end{bmatrix}
\end{array}
\]

\[
\begin{array}{c}
A_2 = \\
\begin{bmatrix}
    r_{11} & r_{12} & \cdots & r_{1n} \\
    r_{21} & r_{22} & \cdots & r_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{m1} & r_{m2} & \cdots & r_{mn}
\end{bmatrix}
\end{array}
\]

\[
\begin{array}{c}
A_3 = \\
\begin{bmatrix}
    r_{10} & r_{20} & \cdots & r_{m0}
\end{bmatrix}
\end{array}
\]

\[
\begin{array}{c}
b_1 = \\
\begin{bmatrix}
    r_{10} & r_{20} & \cdots & r_{m0}
\end{bmatrix}
\end{array}
\]

\[
\begin{array}{c}
c_1 = \\
\begin{bmatrix}
r_{01} & r_{02} & \cdots & r_{0n}
\end{bmatrix}
\end{array}
\]
\[ \frac{\partial H(z_1, z_2)}{\partial r_{kl}} = [c_2 + c_1(z_1 I_m - A_1)^{-1} A_2] \cdot [I_n \ 0] [z_2 I_{2n} - \begin{bmatrix} A_4 & \tilde{e}_1^T \tilde{e}_1^T \\ 0 & A_4 \end{bmatrix}]^{-1} [b_2] \]

\[ \frac{\partial H(z_1, z_2)}{\partial \tau_{00}} = [1 - c_1(z_1 I_m - A_1)^{-1} \alpha] \cdot [1 - \beta(z_2 I_n - A_4)^{-1} b_2] \]

\[ \frac{\partial H(z_1, z_2)}{\partial \tau_{0l}} = c_1(z_1 I_m - A_1)^{-1} \tilde{e}_k [1 - \beta(z_2 I_n - A_4)^{-1} b_2] \]

\[ \frac{\partial H(z_1, z_2)}{\partial \tau_{rl}} = c_1(z_1 I_m - A_1)^{-1} \tilde{e}_k \tilde{e}_l^T (z_2 I_n - A_4)^{-1} b_2 \]

where \( k = 1, 2, \ldots, m, l = 1, 2, \ldots, n \) and

\[ \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}, \quad \beta = [\beta_1 \ \beta_2 \ \cdots \ \beta_n], \quad \tilde{r}_0 = \begin{bmatrix} r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \end{bmatrix} \]

\[ \mathbf{r}_0 = [r_{10} \ r_{20} \ \cdots \ r_{on}] \]

**Definition 2:** Let \( X(z_1, z_2) \) be an \( m \times n \) complex matrix valued function of complex variables \( z_1 \) and \( z_2 \) whose \( k,l \)th element is denoted by \( x_{kl}(z_1, z_2) \). The squared \( L_2 \) norm of \( X(z_1, z_2) \) is then defined as

\[ \|X(z_1, z_2)\|_2^2 = \frac{1}{(2\pi)^2} \int_{|z_1|=1} \int_{|z_2|=1} \sum_{k=1}^{m} \sum_{l=1}^{n} |x_{kl}(z_1, z_2)|^2 d_2 z_2 \]

\[ = \text{tr} \left[ \frac{1}{(2\pi)^2} \int_{|z_1|=1} \int_{|z_2|=1} X(z_1, z_2) X(z_1, z_2)^T d_2 z_1 d_2 z_2 \right] \]

The overall \( L_2 \) sensitivity measure is now defined by

\[ M_\rho = \left\| \frac{\partial H(z_1, z_2)}{\partial \alpha} \right\|_2^2 + \left\| \frac{\partial H(z_1, z_2)}{\partial \beta} \right\|_2^2 + \left\| \frac{\partial H(z_1, z_2)}{\partial \Delta_1} \right\|_2^2 + \sum_{k=1}^{m} \left\| \frac{\partial H(z_1, z_2)}{\partial \Delta_k} \right\|_2^2 + \sum_{k=1}^{m} \left\| \frac{\partial H(z_1, z_2)}{\partial \tau_{00}} \right\|_2^2 + \sum_{k=1}^{m} \sum_{l=1}^{n} \left\| \frac{\partial H(z_1, z_2)}{\partial r_{rl}} \right\|_2^2 \]

From (6), (7) and (8), it follows that

\[ M_\rho = \text{tr} \left[ [I_m \ 0] K_\alpha \left[ \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \right] \right] + \text{tr} \left[ [0 \ I_n] W_\beta \left[ \begin{bmatrix} 0 & I_n \end{bmatrix} \right] \right] + \left[ -c_1 \tilde{e}_1^T \right] K_{\Xi_k} \left[ -c_1 \tilde{e}_1^T \right] + \left[ \tilde{e}_1^T - b_2^T \right] W_{\Delta_1} \left[ -b_2 \right] \]

\[ + \sum_{k=2}^{m} \left[ c_1 \ 0 \right] K_{\Xi_k} \left[ c_1 \ 0 \right] + \sum_{l=1}^{n} \left[ 0 \ b_2^T \right] W_{\delta l} \left[ 0 \ b_2 \right] \]

\[ + \sum_{k=1}^{m} \psi(\tilde{\tau}_k) [c_1 \ 0] K_{\tau_k} [c_1 \ 0] + \sum_{l=1}^{n} \psi(\tilde{r}_l) [0 \ b_2^T] W_{\delta l} [0 \ b_2] \]

\[ + \left( 1 + \alpha^T W_\alpha + \text{tr}[W_\alpha^T \alpha] \right) \left( 1 + \beta K_\nu \beta^T + \text{tr}[K_\nu^T \nu] \right) \]
Remark 1: It is interesting to note that alternatively the $l_2$-sensitivity for generalized direct-form II state-space realization in (3) can be evaluated by

\[ M_{S_2} = \text{tr} \left( K_1 \left[ I_m \ 0 \right] + \sum_{k=2}^{m} c_1 K_1 \left[ 0 \ c_1^T \right] + \sum_{l=2}^{n} b_2 \left[ 0 \ b_2^T \right] W_{\Delta l} \right) \]

(10)

which is essentially identical to $M_{S_2}$ in [20] where

\[
K_1 = \begin{bmatrix} A_1^T & c_1^T \varepsilon_1^T \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} A_1^T & c_1^T \varepsilon_1^T \\ 0 & A_1 \end{bmatrix}^T + \begin{bmatrix} 0 \\ 0 \end{bmatrix} A_2 K^v A_1^T + b_1 b_1^T
\]

\[
W_4 = \begin{bmatrix} A_4 \hat{e}_1 b_1^T \\ 0 \end{bmatrix} \begin{bmatrix} A_4 \hat{e}_1 b_1^T \\ 0 \end{bmatrix} + \begin{bmatrix} A_2^T W^h A_2 + c_2^T c_2 \\ 0 \end{bmatrix}
\]

\[
K^v = A_1 K^v A_1^T + A_2 K^v A_1^T + b_1 b_1^T
\]

\[
W^v = A_1^T W^v A_4 + A_2^T W^h A_2 + c_2^T c_2
\]

From (9) and (10), $\Delta M = M_{S_2} - M_{S_2}$ is given by

\[
\Delta M = \text{tr} \left( K_1 \left[ I_m \ 0 \right] \left( K_\alpha - K_1 \right) \left[ I_m \ 0 \right] \right)
\]

(11)

\[
\Delta M = \text{tr} \left( \begin{bmatrix} 0 \\ I_m \end{bmatrix} \left( W^v - W_4 \right) \left[ I_m \ 0 \right] \right)
\]

(11)

It is noted that the difference $\Delta M = M_{S_2} - M_{S_2}$ in (11) is due to the different number of parameters (coefficients) between the generalized direct-form II structure in (2) and its state-space realization in (3).

IV. A NUMERICAL EXAMPLE

For comparison purposes, we consider the same example as that in [20] where a 2-D stable separable-denominator digital filter of order $(m, n) = (3, 3)$ in (1) with

\[
\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} -2.173645 & 1.836929 & -0.599655 \\ -2.280029 & 1.887939 & -0.564961 \end{bmatrix}
\]

was examined. If $l_2$-scaling constraints were satisfied by choosing $\Delta_k$ and $\Delta_l$ appropriately [20], the filter coefficients in (2) were derived in case $\gamma_k = \gamma_l = 1$ for $k, l = 1, 2, 3$ as

\[
\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix} = \begin{bmatrix} 0.837277 & 0.586915 & 0.166560 \\ 1.084549 & 0.851201 & 0.344142 \end{bmatrix}
\]

\[
\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \end{bmatrix} = \begin{bmatrix} 0.992289 & 0.840742 & 0.457913 \\ 0.663843 & 0.580255 & 0.323990 \end{bmatrix}
\]

\[
\begin{bmatrix} r_{kl} \end{bmatrix} = \begin{bmatrix} 0.1942100 & 0.4600260 & 0.370760 & 0.246578 \\ 0.6359199 & 1.877514 & 1.103626 & 1.163573 \\ 0.7625064 & 2.417902 & 3.332929 & 1.812459 \\ 0.5181348 & 1.905871 & 2.814782 & 5.255824 \end{bmatrix}
\]

TABLE I. PERFORMANCE COMPARISON

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<tr>
<th></th>
<th></th>
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<th></th>
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</thead>
<tbody>
<tr>
<td>Minimized $l_2$-Sensitivity</td>
<td>102.0064</td>
<td>104.6422</td>
<td>98.3040</td>
</tr>
<tr>
<td>Number of Nontrivial Coefficients</td>
<td>40</td>
<td>30</td>
<td>32</td>
</tr>
</tbody>
</table>

From Table I, it is observed that the proposed technique results in the minimum $l_2$-sensitivity among them where there are at most $4(m + n) + mn + 1$ nontrivial coefficients. The method in [7] yields a little larger $l_2$-sensitivity and its filter implementation is costly where $m^2 + mn + n^2 + 2(m + n) + 1$ nontrivial coefficients exist. The filter obtained by [20] is less expensive in terms of the filter implementation cost, but it has larger $l_2$-sensitivity where at most $4(m + n) + mn - 1$ nontrivial coefficients exist in the state-space realization of (3).

V. CONCLUSION

A new expression of evaluating $l_2$-sensitivity for generalized direct-form II structure (described by Eq. (2) or illustrated in Fig. 2) of 2-D separable-denominator digital filters has been derived and analyzed. The $l_2$-sensitivity measure has been compared with that deduced in a recent study of generalized direct-form II state-space realization (described by Eq. (3)) of 2-D separable-denominator digital filters [20]. In a numerical example, the $l_2$-sensitivity measure have been minimized with respect to the free parameters subject to $l_2$-scaling constraints by exhaustive search in a finite element space. Finally, the numerical results have been compared with those obtained by the techniques in [7] and [20].
REFERENCES


