On l_2-Sensitivity for Generalized Direct-Form II Structure of 2-D Separable-Denominator Filters

Takao Hinamoto
Hiroshima University
Higashi-Hiroshima 739-8527, Japan
Email: hinamoto@ieee.org

Akimitsu Doi
Hiroshima Institute of Technology
Hiroshima 731-5193, Japan
Email: doi@cc.it-hiroshima.ac.jp

Wu-Sheng Lu
University of Victoria
Victoria, BC, Canada V8W 3P6
Email: wslu@ece.uvic.ca

***Abstract*—A new expression of evaluating l_2-sensitivity for generalized direct-form II structure of two-dimensional (2-D) separable-denominator digital filters is derived and analyzed. Unlike the previous work, l_2-sensitivity is analyzed for the filter structure instead of its state-space realization. Then the resulting l_2-sensitivity measure is compared with that deduced in a recent study of generalized direct-form II state-space realization of 2-D separable in denominator digital filters. In a numerical example, the new l_2-sensitivity measure is minimized with respect to free parameters subject to l_2-scaling constraints by an exhaustive search over a set of discrete values of finite cardinality, and the results are compared with those obtained by existing techniques.**

I. Introduction

2-D separable-denominator digital filters are recognized as a very popular class of multi-dimensional signal processors due to the need in realizing a 2-D magnitude response with various symmetries of quarter-plane filters except for diagonal symmetry, easy stability test performed by simply calculating the poles of the two one-dimensional polynomials, and fewer multipliers required to realize the transfer function, especially in the case of symmetric magnitude response [1]-[4]. It is well known that the reduction of coefficient sensitivities is an important issue in the implementation of IIR digital filters using fixed-point arithmetic. For 2-D state-space digital filters whose transfer functions are separable in denominator, coefficient sensitivities have been analyzed using either a mixture of l_1/l_2 norms [5] or a pure l_2 norm [6],[7]. 2-D state-space filter structures with low coefficient sensitivities have also been synthesized [5]-[7]. Although the l_2-sensitivity minimization is technically more challenging, it is more natural and reasonable than the conventional l_1/l_2 mixed sensitivity minimization. Alternatively, delta-operator-based implementations have received considerable interest because of their excellent finite-word-length performance under fast sampling, and delta-operator-based designs have extensively studied in the area of digital signal processing and digital control systems[8]-[17]. Recently, generalized direct-form II *structure* and its *state-space realization* of 2-D separable-denominator digital filters have been presented [18]. In addition, the analysis and minimization of *round-off noise* have been explored [18],[19]. More recently, the problem of analyzing and minimizing the l_2-*sensitivity* for generalized direct-form II *state-space realization* of 2-D separable-denominator digital filters has been considered and investigated [20].

In this paper, we analyze the l_2-*sensitivity* for generalized direct-form II *structure* of 2-D separable-denominator digital filters. In [20], l_2-*sensitivity* analysis was performed not for the filter structure, but its state-space realization. The resulting l_2-*sensitivity* measure is then compared with *that* deduced in a recent study of generalized direct-form II *state-space realization* of 2-D separable-denominator digital filters [20]. In a numerical example, the new l_2-*sensitivity* measure is minimized with respect to the free parameters subject to l_2-scaling constraints by exhaustive search in a finite element space. The numerical results are compared with those obtained by the techniques in [7] and [20].

II. Structure of 2-D Digital Filters

Consider a stable 2-D separable-denominator digital filter described by

$$H(z_1,z_2)=\frac{\displaystyle\sum_{k=0}^{m}\sum_{l=0}^{n}c_{kl}z_1^{m-k}z_2^{n-l}}{\displaystyle\Big(z_1^m+\sum_{k=1}^{m}a_kz_1^{m-k}\Big)\Big(z_2^n+\sum_{l=1}^{n}b_lz_2^{n-l}\Big)} \tag{1}$$

where the denominator and numerator are assumed coprime. It was shown [18] that the transfer function in (1) can be converted into

$$H(z_1,z_2)=\frac{\displaystyle\sum_{k=0}^{m}\sum_{l=0}^{n}r_{kl}\prod_{p=0}^{k}\rho_p^h(z_1)^{-1}\prod_{q=0}^{l}\rho_q^v(z_2)^{-1}}{\displaystyle\Big(\sum_{k=0}^{m}\alpha_k\prod_{p=0}^{k}\rho_p^h(z_1)^{-1}\Big)\Big(\sum_{l=0}^{n}\beta_l\prod_{q=0}^{l}\rho_q^v(z_2)^{-1}\Big)} \tag{2}$$

where $\alpha_0=\beta_0=1$ and $\rho_0^h(z_1)^{-1}=\rho_0^v(z_2)^{-1}=1$. The implementations of generalized delta operators $\rho_k^h(z_1)^{-1}$ and $\rho_l^v(z_2)^{-1}$ are depicted in Fig. 1. As an example, the generalized filter structure in (2) with $(m,n)=(3,3)$ is illustrated in Fig. 2 where $u(i,j)$ is a scalar input and $y(i,j)$ is a scalar output.

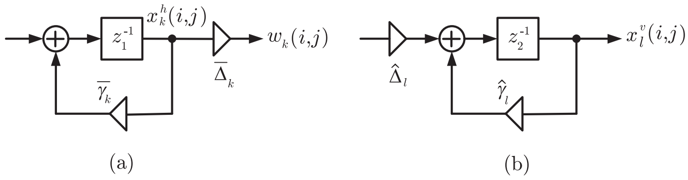

Fig. 1. (a) Implementation of $\rho_k^h(z_1)^{-1}$ and (b) that of $\rho_l^v(z_2)^{-1}$.

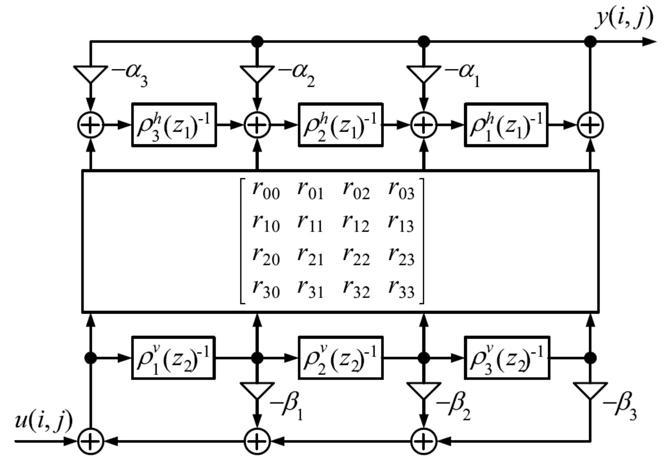

Fig. 2. The 2-D filter structure of (2) with $(m,n)=(3,3)$.

As is shown in [18], the 2-D filter in (2) can be realized by the Roesser model $(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{A}_4,\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{c}_1,\boldsymbol{c}_2,d)_{m,n}$ [21] as

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{0} & \boldsymbol{A}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} u(i,j) \\
y(i,j) &= [\,\boldsymbol{c}_1 \quad \boldsymbol{c}_2\,] \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + d\,u(i,j)
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
\boldsymbol{x}^h(i,j) &= [\, x_1^h(i,j) \quad x_2^h(i,j) \quad \cdots \quad x_m^h(i,j) \,]^T \\
\boldsymbol{x}^v(i,j) &= [\, x_1^v(i,j) \quad x_2^v(i,j) \quad \cdots \quad x_n^v(i,j) \,]^T
\end{aligned}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} -\alpha_1\overline{\Delta}_1 & \overline{\Delta}_2 & \cdots & 0 \\ -\alpha_2\overline{\Delta}_1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \overline{\Delta}_m \\ -\alpha_m\overline{\Delta}_1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} \overline{\gamma}_1 & 0 & \cdots & 0 \\ 0 & \overline{\gamma}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \overline{\gamma}_m \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{A}_2 &= \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} - \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} [r_{01} \quad r_{02} \quad \cdots \quad r_{0n}] \\
&\quad - \begin{bmatrix} r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \end{bmatrix} [\beta_1 \ \beta_2 \cdots \beta_n] + r_{00} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} [\beta_1 \ \beta_2 \cdots \beta_n]
\end{aligned}$$

$$\boldsymbol{A}_4 = \begin{bmatrix} -\beta_1\hat{\Delta}_1 & -\beta_2\hat{\Delta}_1 & \cdots & -\beta_n\hat{\Delta}_1 \\ \hat{\Delta}_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \hat{\Delta}_n & 0 \end{bmatrix} + \begin{bmatrix} \hat{\gamma}_1 & 0 & \cdots & 0 \\ 0 & \hat{\gamma}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \hat{\gamma}_n \end{bmatrix}$$

$$\boldsymbol{b}_1 = \begin{bmatrix} r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \end{bmatrix} - r_{00} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} \hat{\Delta}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{c}_1 &= [\, \overline{\Delta}_1 \quad 0 \quad \cdots \quad 0 \,] \\
\boldsymbol{c}_2 &= [r_{01} \quad r_{02} \quad \cdots \quad r_{0n}] - r_{00}\,[\beta_1 \ \beta_2 \cdots \beta_n], \qquad d = r_{00}
\end{aligned}$$

The transfer function of the filter in (3) is given by [22]

$$\begin{aligned}
H(z_1,z_2) &= [\,\boldsymbol{c}_1 \quad \boldsymbol{c}_2\,] \begin{bmatrix} z_1\boldsymbol{I}_m - \boldsymbol{A}_1 & -\boldsymbol{A}_2 \\ \boldsymbol{0} & z_2\boldsymbol{I}_n - \boldsymbol{A}_4 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} + d \\
&= \boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{b}_1 + \boldsymbol{c}_2(z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\boldsymbol{b}_2 \\
&\quad + \boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_2(z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\boldsymbol{b}_2 + d
\end{aligned} \tag{4}$$

III. Novel Analysis of l_2-Sensitivity

We are now in a position to analyze the l_2-sensitivity for the generalized direct-form II structure of 2-D separable in denominator digital filters in (2) with $4(m+n)+mn+1$ coefficients. It is noted that l_2-sensitivity for its state-space realization in (3) has already been analyzed in [20].

Definition 1: Let $\boldsymbol{X}$ be an $m \times n$ real matrix and $f(\boldsymbol{X})$ be a scalar complex function of $\boldsymbol{X}$, differentiable with respect to all the entries of $\boldsymbol{X}$. The sensitivity function of $f(\boldsymbol{X})$ with respect to $\boldsymbol{X}$ is then defined as

$$\boldsymbol{S}_X = \frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}} \quad \text{with} \quad (\boldsymbol{S}_X)_{kl} = \frac{\partial f(\boldsymbol{X})}{\partial x_{kl}} \tag{5}$$

where x_{kl} denotes the (k,l)th element of a matrix $\boldsymbol{X}$.

It can be shown that

$$\begin{aligned}
\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{\alpha}} &= -[\,\boldsymbol{I}_m \quad \boldsymbol{0}\,]\left[z_1\boldsymbol{I}_{2m} - \begin{bmatrix} \boldsymbol{A}_1^T & \boldsymbol{c}_1^T\boldsymbol{c}_1 \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix}\right]^{-1} \\
&\quad \cdot \begin{bmatrix} r_{00}\boldsymbol{c}_1^T & \boldsymbol{c}_1^T(\overline{\boldsymbol{r}}_0 - r_{00}\boldsymbol{\beta}) \\ \boldsymbol{b}_1 & \boldsymbol{A}_2 \end{bmatrix} \begin{bmatrix} 1 \\ (z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\boldsymbol{b}_2 \end{bmatrix} \\
\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{\beta}} &= -\left[\,1 \quad \boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\,\right] \\
&\quad \cdot \begin{bmatrix} \boldsymbol{c}_2 & r_{00}\boldsymbol{b}_2^T \\ \boldsymbol{A}_2 & (\hat{\boldsymbol{r}}_0 - r_{00}\boldsymbol{\alpha})\boldsymbol{b}_2^T \end{bmatrix} \left[z_2\boldsymbol{I}_{2n} - \begin{bmatrix} \boldsymbol{A}_4 & \boldsymbol{b}_2\boldsymbol{b}_2^T \\ \boldsymbol{0} & \boldsymbol{A}_4^T \end{bmatrix}\right]^{-1} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_n \end{bmatrix} \\
\frac{\partial H(z_1,z_2)}{\partial \overline{\Delta}_1} &= \left[-\boldsymbol{c}_1 \quad \overline{\boldsymbol{e}}_1^T\,\right]\left[z_1\boldsymbol{I}_{2m} - \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{\alpha}\overline{\boldsymbol{e}}_1^T \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix}\right]^{-1} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{bmatrix} \\
&\quad \cdot [\boldsymbol{b}_1 + \boldsymbol{A}_2(z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\boldsymbol{b}_2] \\
\frac{\partial H(z_1,z_2)}{\partial \hat{\Delta}_1} &= [\boldsymbol{c}_2 + \boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_2] \\
&\quad \cdot [\,\boldsymbol{I}_n \quad \boldsymbol{0}\,]\left[z_2\boldsymbol{I}_{2n} - \begin{bmatrix} \boldsymbol{A}_4 & \hat{\boldsymbol{e}}_1\boldsymbol{\beta} \\ \boldsymbol{0} & \boldsymbol{A}_4 \end{bmatrix}\right]^{-1} \begin{bmatrix} \hat{\boldsymbol{e}}_1 \\ -\boldsymbol{b}_2 \end{bmatrix} \\
\frac{\partial H(z_1,z_2)}{\partial \overline{\Delta}_k} &= [\,\boldsymbol{c}_1 \quad \boldsymbol{0}\,]\left[z_1\boldsymbol{I}_{2m} - \begin{bmatrix} \boldsymbol{A}_1 & \overline{\boldsymbol{e}}_{k-1}\overline{\boldsymbol{e}}_k^T \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix}\right]^{-1} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{bmatrix} \\
(k \geq 2) \quad &\quad \cdot [\boldsymbol{b}_1 + \boldsymbol{A}_2(z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\boldsymbol{b}_2] \\
\frac{\partial H(z_1,z_2)}{\partial \hat{\Delta}_l} &= [\boldsymbol{c}_2 + \boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_2] \\
(l \geq 2) \quad &\quad \cdot [\,\boldsymbol{I}_n \quad \boldsymbol{0}\,]\left[z_2\boldsymbol{I}_{2n} - \begin{bmatrix} \boldsymbol{A}_4 & \hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_{l-1}^T \\ \boldsymbol{0} & \boldsymbol{A}_4 \end{bmatrix}\right]^{-1} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b}_2 \end{bmatrix} \\
\frac{\partial H(z_1,z_2)}{\partial \overline{\gamma}_k} &= [\,\boldsymbol{c}_1 \quad \boldsymbol{0}\,]\left[z_1\boldsymbol{I}_{2m} - \begin{bmatrix} \boldsymbol{A}_1 & \overline{\boldsymbol{e}}_k\overline{\boldsymbol{e}}_k^T \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix}\right]^{-1} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{bmatrix} \\
&\quad \cdot [\boldsymbol{b}_1 + \boldsymbol{A}_2(z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\boldsymbol{b}_2]
\end{aligned} \tag{6}$$

$$
\begin{aligned}
\frac{\partial H(z_1,z_2)}{\partial \hat{\gamma}_l} &= [\boldsymbol{c}_2+\boldsymbol{c}_1(z_1\boldsymbol{I}_m-\boldsymbol{A}_1)^{-1}\boldsymbol{A}_2] \\
&\quad \cdot [\,\boldsymbol{I}_n \;\; \boldsymbol{0}\,]\left[z_2\boldsymbol{I}_{2n}-\begin{bmatrix}\boldsymbol{A}_4 & \hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_l^T\\ \boldsymbol{0} & \boldsymbol{A}_4\end{bmatrix}\right]^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{b}_2\end{bmatrix} \\
\frac{\partial H(z_1,z_2)}{\partial r_{00}} &= [1-\boldsymbol{c}_1(z_1\boldsymbol{I}_m-\boldsymbol{A}_1)^{-1}\boldsymbol{\alpha}] \\
&\qquad \cdot[1-\boldsymbol{\beta}(z_2\boldsymbol{I}_n-\boldsymbol{A}_4)^{-1}\boldsymbol{b}_2] \\
\frac{\partial H(z_1,z_2)}{\partial r_{k0}} &= \boldsymbol{c}_1(z_1\boldsymbol{I}_m-\boldsymbol{A}_1)^{-1}\overline{\boldsymbol{e}}_k[1-\boldsymbol{\beta}(z_2\boldsymbol{I}_n-\boldsymbol{A}_4)^{-1}\boldsymbol{b}_2] \\
\frac{\partial H(z_1,z_2)}{\partial r_{0l}} &= [1-\boldsymbol{c}_1(z_1\boldsymbol{I}_m-\boldsymbol{A}_1)^{-1}\boldsymbol{\alpha}]\hat{\boldsymbol{e}}_l^T(z_2\boldsymbol{I}_n-\boldsymbol{A}_4)^{-1}\boldsymbol{b}_2 \\
\frac{\partial H(z_1,z_2)}{\partial r_{kl}} &= \boldsymbol{c}_1(z_1\boldsymbol{I}_m-\boldsymbol{A}_1)^{-1}\overline{\boldsymbol{e}}_k\hat{\boldsymbol{e}}_l^T(z_2\boldsymbol{I}_n-\boldsymbol{A}_4)^{-1}\boldsymbol{b}_2
\end{aligned}
$$

where $k=1,2,\cdots,m$, $l=1,2,\cdots,n$ and

$$
\boldsymbol{\alpha}=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{bmatrix},\quad \boldsymbol{\beta}=[\,\beta_1\;\;\beta_2\;\;\cdots\;\;\beta_n\,],\quad \hat{\boldsymbol{r}}_0=\begin{bmatrix}r_{10}\\ r_{20}\\ \vdots\\ r_{m0}\end{bmatrix}
$$

$$
\overline{\boldsymbol{r}}_0=[\,r_{01}\;\;r_{02}\;\;\cdots\;\;r_{0n}\,]
$$

Definition 2: Let $\boldsymbol{X}(z_1,z_2)$ be an $m\times n$ complex matrix valued function of complex variables z_1 and z_2 whose (k,l)th element is denoted by $x_{kl}(z_1,z_2)$. The squared l_2-norm of $\boldsymbol{X}(z_1,z_2)$ is then defined as

$$
\begin{aligned}
&||\boldsymbol{X}(z_1,z_2)||_2^2 \\
&= \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\sum_{k=1}^{m}\sum_{l=1}^{n}|x_{kl}(z_1,z_2)|^2\,\frac{dz_1dz_2}{z_1z_2} \\
&= \mathrm{tr}\left[\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\boldsymbol{X}(z_1,z_2)\boldsymbol{X}(z_1,z_2)^H\frac{dz_1dz_2}{z_1z_2}\right]
\end{aligned}
\tag{7}
$$

The overall L_2-sensitivity measure is now defined by

$$
\begin{aligned}
M_\rho = &\left|\left|\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{\alpha}}\right|\right|_2^2+\left|\left|\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{\beta}}\right|\right|_2^2+\left|\left|\frac{\partial H(z_1,z_2)}{\partial \overline{\Delta}_1}\right|\right|_2^2 \\
&+\left|\left|\frac{\partial H(z_1,z_2)}{\partial \hat{\Delta}_1}\right|\right|_2^2+\sum_{k=2}^{m}\left|\left|\frac{\partial H(z_1,z_2)}{\partial \overline{\Delta}_k}\right|\right|_2^2 \\
&+\sum_{l=2}^{n}\left|\left|\frac{\partial H(z_1,z_2)}{\partial \hat{\Delta}_l}\right|\right|_2^2+\sum_{k=1}^{m}\psi(\overline{\gamma}_k)\left|\left|\frac{\partial H(z_1,z_2)}{\partial \overline{\gamma}_k}\right|\right|_2^2 \\
&+\sum_{l=1}^{n}\psi(\hat{\gamma}_l)\left|\left|\frac{\partial H(z_1,z_2)}{\partial \hat{\gamma}_l}\right|\right|_2^2+\left|\left|\frac{\partial H(z_1,z_2)}{\partial r_{00}}\right|\right|_2^2 \\
&+\sum_{k=1}^{m}\sum_{l=1}^{n}\left|\left|\frac{\partial H(z_1,z_2)}{\partial r_{kl}}\right|\right|_2^2+\sum_{k=1}^{m}\left|\left|\frac{\partial H(z_1,z_2)}{\partial r_{k0}}\right|\right|_2^2 \\
&+\sum_{l=1}^{n}\left|\left|\frac{\partial H(z_1,z_2)}{\partial r_{0l}}\right|\right|_2^2
\end{aligned}
\tag{8}
$$

where

$$
\psi(x)=\begin{cases}1 & \text{for}\quad x\neq 0\ ,\pm 1\\ 0 & \text{for}\quad x=0\ ,\pm 1\end{cases}
$$

From (6), (7) and (8), it follows that

$$
\begin{aligned}
M_\rho = &\,\mathrm{tr}\left[[\,\boldsymbol{I}_m\;\;\boldsymbol{0}\,]\boldsymbol{K}_\alpha\begin{bmatrix}\boldsymbol{I}_m\\ \boldsymbol{0}\end{bmatrix}\right]+\mathrm{tr}\left[[\,\boldsymbol{0}\;\;\boldsymbol{I}_n\,]\boldsymbol{W}_\beta\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}_n\end{bmatrix}\right] \\
&+[-\boldsymbol{c}_1\;\;\overline{\boldsymbol{e}}_1^T]\,\boldsymbol{K}_{\overline{\Delta}_1}\begin{bmatrix}-\boldsymbol{c}_1^T\\ \overline{\boldsymbol{e}}_1\end{bmatrix}+[\,\hat{\boldsymbol{e}}_1^T\;\;-\boldsymbol{b}_2^T]\,\boldsymbol{W}_{\hat{\Delta}_1}\begin{bmatrix}\hat{\boldsymbol{e}}_1\\ -\boldsymbol{b}_2\end{bmatrix} \\
&+\sum_{k=2}^{m}[\,\boldsymbol{c}_1\;\;\boldsymbol{0}\,]\,\boldsymbol{K}_{\overline{\Delta}_k}\begin{bmatrix}\boldsymbol{c}_1^T\\ \boldsymbol{0}\end{bmatrix}+\sum_{l=2}^{n}[\,\boldsymbol{0}\;\;\boldsymbol{b}_2^T\,]\,\boldsymbol{W}_{\hat{\Delta}_l}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{b}_2\end{bmatrix} \\
&+\sum_{k=1}^{m}\psi(\overline{\gamma}_k)\,[\,\boldsymbol{c}_1\;\;\boldsymbol{0}]\boldsymbol{K}_{\overline{\gamma}_k}\begin{bmatrix}\boldsymbol{c}_1^T\\ \boldsymbol{0}\end{bmatrix}+\sum_{l=1}^{n}\psi(\hat{\gamma}_l)\,[\,\boldsymbol{0}\;\;\boldsymbol{b}_2^T\,]\boldsymbol{W}_{\hat{\gamma}_l}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{b}_2\end{bmatrix} \\
&+\Big(1+\boldsymbol{\alpha}^T\boldsymbol{W}^h\boldsymbol{\alpha}+\mathrm{tr}[\boldsymbol{W}^h]\Big)\Big(1+\boldsymbol{\beta}\boldsymbol{K}^v\boldsymbol{\beta}^T+\mathrm{tr}[\boldsymbol{K}^v]\Big)
\end{aligned}
\tag{9}
$$

where the Grammians $\boldsymbol{K}_\alpha$, $\boldsymbol{W}_\beta$, etc. can be obtained by solving the following Lyapunov equations:

$$
\begin{aligned}
\boldsymbol{K}_\alpha &= \begin{bmatrix}\boldsymbol{A}_1^T & \boldsymbol{c}_1^T\boldsymbol{c}_1\\ \boldsymbol{0} & \boldsymbol{A}_1\end{bmatrix}\boldsymbol{K}_\alpha\begin{bmatrix}\boldsymbol{A}_1^T & \boldsymbol{c}_1^T\boldsymbol{c}_1\\ \boldsymbol{0} & \boldsymbol{A}_1\end{bmatrix}^T \\
&+\begin{bmatrix}r_{00}\boldsymbol{c}_1^T & \boldsymbol{c}_1^T(\overline{\boldsymbol{r}}_0-r_{00}\boldsymbol{\beta})\\ \boldsymbol{b}_1 & \boldsymbol{A}_2\end{bmatrix}\begin{bmatrix}1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}^v\end{bmatrix}\begin{bmatrix}r_{00}\boldsymbol{c}_1^T & \boldsymbol{c}_1^T(\overline{\boldsymbol{r}}_0-r_{00}\boldsymbol{\beta})\\ \boldsymbol{b}_1 & \boldsymbol{A}_2\end{bmatrix}^T \\
\boldsymbol{W}_\beta &= \begin{bmatrix}\boldsymbol{A}_4 & \boldsymbol{b}_2\boldsymbol{b}_2^T\\ \boldsymbol{0} & \boldsymbol{A}_4^T\end{bmatrix}^T\boldsymbol{W}_\beta\begin{bmatrix}\boldsymbol{A}_4 & \boldsymbol{b}_2\boldsymbol{b}_2^T\\ \boldsymbol{0} & \boldsymbol{A}_4^T\end{bmatrix} \\
&+\begin{bmatrix}\boldsymbol{c}_2 & r_{00}\boldsymbol{b}_2^T\\ \boldsymbol{A}_2 & (\hat{\boldsymbol{r}}_0-r_{00}\boldsymbol{\alpha})\boldsymbol{b}_2^T\end{bmatrix}^T\begin{bmatrix}1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{W}^h\end{bmatrix}\begin{bmatrix}\boldsymbol{c}_2 & r_{00}\boldsymbol{b}_2^T\\ \boldsymbol{A}_2 & (\hat{\boldsymbol{r}}_0-r_{00}\boldsymbol{\alpha})\boldsymbol{b}_2^T\end{bmatrix} \\
\boldsymbol{K}^v &= \boldsymbol{A}_4\boldsymbol{K}^v\boldsymbol{A}_4^T+\boldsymbol{b}_2\boldsymbol{b}_2^T,\qquad \boldsymbol{W}^h=\boldsymbol{A}_1^T\boldsymbol{W}^h\boldsymbol{A}_1+\boldsymbol{c}_1^T\boldsymbol{c}_1 \\
\boldsymbol{K}_{\overline{\Delta}_1} &= \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{\alpha}\overline{\boldsymbol{e}}_1^T\\ \boldsymbol{0} & \boldsymbol{A}_1\end{bmatrix}\boldsymbol{K}_{\overline{\Delta}_1}\begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{\alpha}\overline{\boldsymbol{e}}_1^T\\ \boldsymbol{0} & \boldsymbol{A}_1\end{bmatrix}^T \\
&+\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}_2\boldsymbol{K}^v\boldsymbol{A}_2^T+\boldsymbol{b}_1\boldsymbol{b}_1^T\end{bmatrix} \\
\boldsymbol{W}_{\hat{\Delta}_1} &= \begin{bmatrix}\boldsymbol{A}_4 & \hat{\boldsymbol{e}}_1\boldsymbol{\beta}\\ \boldsymbol{0} & \boldsymbol{A}_4\end{bmatrix}^T\boldsymbol{W}_{\hat{\Delta}_1}\begin{bmatrix}\boldsymbol{A}_4 & \hat{\boldsymbol{e}}_1\boldsymbol{\beta}\\ \boldsymbol{0} & \boldsymbol{A}_4\end{bmatrix} \\
&+\begin{bmatrix}\boldsymbol{A}_2^T\boldsymbol{W}^h\boldsymbol{A}_2+\boldsymbol{c}_2^T\boldsymbol{c}_2 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \\
\boldsymbol{K}_{\overline{\Delta}_k} &= \begin{bmatrix}\boldsymbol{A}_1 & \overline{\boldsymbol{e}}_{k-1}\overline{\boldsymbol{e}}_k^T\\ \boldsymbol{0} & \boldsymbol{A}_1\end{bmatrix}\boldsymbol{K}_{\overline{\Delta}_k}\begin{bmatrix}\boldsymbol{A}_1 & \overline{\boldsymbol{e}}_{k-1}\overline{\boldsymbol{e}}_k^T\\ \boldsymbol{0} & \boldsymbol{A}_1\end{bmatrix}^T \\
&+\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}_2\boldsymbol{K}^v\boldsymbol{A}_2^T+\boldsymbol{b}_1\boldsymbol{b}_1^T\end{bmatrix} \\
\boldsymbol{W}_{\hat{\Delta}_l} &= \begin{bmatrix}\boldsymbol{A}_4 & \hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_{l-1}^T\\ \boldsymbol{0} & \boldsymbol{A}_4\end{bmatrix}^T\boldsymbol{W}_{\hat{\Delta}_l}\begin{bmatrix}\boldsymbol{A}_4 & \hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_{l-1}^T\\ \boldsymbol{0} & \boldsymbol{A}_4\end{bmatrix} \\
&+\begin{bmatrix}\boldsymbol{A}_2^T\boldsymbol{W}^h\boldsymbol{A}_2+\boldsymbol{c}_2^T\boldsymbol{c}_2 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \\
\boldsymbol{K}_{\overline{\gamma}_k} &= \begin{bmatrix}\boldsymbol{A}_1 & \overline{\boldsymbol{e}}_k\overline{\boldsymbol{e}}_k^T\\ \boldsymbol{0} & \boldsymbol{A}_1\end{bmatrix}\boldsymbol{K}_{\overline{\gamma}_k}\begin{bmatrix}\boldsymbol{A}_1 & \overline{\boldsymbol{e}}_k\overline{\boldsymbol{e}}_k^T\\ \boldsymbol{0} & \boldsymbol{A}_1\end{bmatrix}^T \\
&+\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}_2\boldsymbol{K}^v\boldsymbol{A}_2^T+\boldsymbol{b}_1\boldsymbol{b}_1^T\end{bmatrix} \\
\boldsymbol{W}_{\hat{\gamma}_l} &= \begin{bmatrix}\boldsymbol{A}_4 & \hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_l^T\\ \boldsymbol{0} & \boldsymbol{A}_4\end{bmatrix}^T\boldsymbol{W}_{\hat{\gamma}_l}\begin{bmatrix}\boldsymbol{A}_4 & \hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_l^T\\ \boldsymbol{0} & \boldsymbol{A}_4\end{bmatrix} \\
&+\begin{bmatrix}\boldsymbol{A}_2^T\boldsymbol{W}^h\boldsymbol{A}_2+\boldsymbol{c}_2^T\boldsymbol{c}_2 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}
\end{aligned}
$$

Remark 1: It is interesting to note that alternatively the l_2-sensitivity for generalized direct-form II state-space realization in (3) can be evaluated by

$$
\begin{aligned}
M_{S\rho} = \ & \mathrm{tr}\left[[\, \boldsymbol{I}_m \ \ \boldsymbol{0} \,] \boldsymbol{K}_1 \begin{bmatrix} \boldsymbol{I}_m \\ \boldsymbol{0} \end{bmatrix} \right] + \mathrm{tr}\left[[\, \boldsymbol{0} \ \ \boldsymbol{I}_n \,] \boldsymbol{W}_4 \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_n \end{bmatrix} \right] \\
& + \sum_{k=2}^{m} [\, \boldsymbol{c}_1 \ \ \boldsymbol{0} \,] \boldsymbol{K}_{\overline{\Delta}_k} \begin{bmatrix} \boldsymbol{c}_1^T \\ \boldsymbol{0} \end{bmatrix} + \sum_{l=2}^{n} [\, \boldsymbol{0} \ \ \boldsymbol{b}_2^T \,] \boldsymbol{W}_{\hat{\Delta}_l} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b}_2 \end{bmatrix} \\
& + \sum_{k=2}^{m} \psi(\overline{\gamma}_k) [\, \boldsymbol{c}_1 \ \ \boldsymbol{0} \,] \boldsymbol{K}_{\overline{\gamma}_k} \begin{bmatrix} \boldsymbol{c}_1^T \\ \boldsymbol{0} \end{bmatrix} + \overline{\boldsymbol{e}}_1^T \boldsymbol{K}^h \overline{\boldsymbol{e}}_1 \\
& + \sum_{l=2}^{n} \psi(\hat{\gamma}_l) [\, \boldsymbol{0} \ \ \boldsymbol{b}_2^T \,] \boldsymbol{W}_{\hat{\gamma}_l} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b}_2 \end{bmatrix} + \hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 \\
& + \big(1 + \mathrm{tr}[\boldsymbol{W}^h]\big)\big(1 + \mathrm{tr}[\boldsymbol{K}^v]\big) + \psi(r_{00}) - 1
\end{aligned} \tag{10}
$$

which is essentially identical to $M_{S\rho}$ in [20] where

$$
\begin{aligned}
\boldsymbol{K}_1 &= \begin{bmatrix} \boldsymbol{A}_1^T & \boldsymbol{c}_1^T \overline{\boldsymbol{e}}_1^T \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix} \boldsymbol{K}_1 \begin{bmatrix} \boldsymbol{A}_1^T & \boldsymbol{c}_1^T \overline{\boldsymbol{e}}_1^T \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix}^T + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_2 \boldsymbol{K}^v \boldsymbol{A}_2^T + \boldsymbol{b}_1 \boldsymbol{b}_1^T \end{bmatrix} \\
\boldsymbol{W}_4 &= \begin{bmatrix} \boldsymbol{A}_4 & \hat{\boldsymbol{e}}_1 \boldsymbol{b}_2^T \\ \boldsymbol{0} & \boldsymbol{A}_4^T \end{bmatrix}^T \boldsymbol{W}_4 \begin{bmatrix} \boldsymbol{A}_4 & \hat{\boldsymbol{e}}_1 \boldsymbol{b}_2^T \\ \boldsymbol{0} & \boldsymbol{A}_4^T \end{bmatrix} + \begin{bmatrix} \boldsymbol{A}_2^T \boldsymbol{W}^h \boldsymbol{A}_2 + \boldsymbol{c}_2^T \boldsymbol{c}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\
\boldsymbol{K}^h &= \boldsymbol{A}_1 \boldsymbol{K}^h \boldsymbol{A}_1^T + \boldsymbol{A}_2 \boldsymbol{K}^v \boldsymbol{A}_2^T + \boldsymbol{b}_1 \boldsymbol{b}_1^T \\
\boldsymbol{W}^v &= \boldsymbol{A}_4^T \boldsymbol{W}^v \boldsymbol{A}_4 + \boldsymbol{A}_2^T \boldsymbol{W}^h \boldsymbol{A}_2 + \boldsymbol{c}_2^T \boldsymbol{c}_2
\end{aligned}
$$

From (9) and (10), $\Delta M = M_\rho - M_{S\rho}$ is given by

$$
\begin{aligned}
\Delta M = \ & \mathrm{tr}\left[[\, \boldsymbol{I}_m \ \ \boldsymbol{0} \,] (\boldsymbol{K}_\alpha - \boldsymbol{K}_1) \begin{bmatrix} \boldsymbol{I}_m \\ \boldsymbol{0} \end{bmatrix} \right] \\
& + \mathrm{tr}\left[[\, \boldsymbol{0} \ \ \boldsymbol{I}_n \,] (\boldsymbol{W}_\beta - \boldsymbol{W}_4) \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_n \end{bmatrix} \right] \\
& + [\, -\boldsymbol{c}_1 \ \ \overline{\boldsymbol{e}}_1^T \,] \boldsymbol{K}_{\overline{\Delta}_1} \begin{bmatrix} -\boldsymbol{c}_1^T \\ \overline{\boldsymbol{e}}_1 \end{bmatrix} + [\, \hat{\boldsymbol{e}}_1^T \ \ -\boldsymbol{b}_2^T \,] \boldsymbol{W}_{\hat{\Delta}_1} \begin{bmatrix} \hat{\boldsymbol{e}}_1 \\ -\boldsymbol{b}_2 \end{bmatrix} \\
& + \psi(\overline{\gamma}_1) [\, \boldsymbol{c}_1 \ \ \boldsymbol{0} \,] \boldsymbol{K}_{\overline{\gamma}_1} \begin{bmatrix} \boldsymbol{c}_1^T \\ \boldsymbol{0} \end{bmatrix} + \psi(\hat{\gamma}_1) [\, \boldsymbol{0} \ \ \boldsymbol{b}_2^T \,] \boldsymbol{W}_{\hat{\gamma}_1} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b}_2 \end{bmatrix} \\
& + (1 + \boldsymbol{\alpha}^T \boldsymbol{W}^h \boldsymbol{\alpha})(1 + \boldsymbol{\beta} \boldsymbol{K}^v \boldsymbol{\beta}^T) + \boldsymbol{\alpha}^T \boldsymbol{W}^h \boldsymbol{\alpha} \, \mathrm{tr}[\boldsymbol{K}^v] \\
& + \boldsymbol{\beta} \boldsymbol{K}^v \boldsymbol{\beta}^T \mathrm{tr}[\boldsymbol{W}^h] - \overline{\boldsymbol{e}}_1^T \boldsymbol{K}^h \overline{\boldsymbol{e}}_1 - \hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 - \psi(r_{00})
\end{aligned} \tag{11}
$$

It is noted that the difference $\Delta M = M_\rho - M_{S\rho}$ in (11) is due to the different number of *parameters (coefficients)* between the generalized direct-form II *structure* in (2) and its *state-space realization* in (3).

IV. A Numerical Example

For comparison purposes, we consider the same example as that in [20] where a 2-D stable separable-denominator digital filter of order $(m, n) = (3, 3)$ in (1) with

$$
\begin{aligned}
[\, a_1 \ \ a_2 \ \ a_3 \,] &= [\, -2.173645 \ \ 1.836929 \ \ -0.599655 \,] \\
[\, b_1 \ \ b_2 \ \ b_3 \,] &= [\, -2.280029 \ \ 1.887939 \ \ -0.564961 \,]
\end{aligned}
$$

$$
[\, c_{kl} \,] = \begin{bmatrix} 0.019421 & -0.027724 & 0.011468 & -0.000087 \\ 0.004839 & 0.017545 & -0.050267 & 0.033061 \\ -0.004328 & -0.008847 & 0.096260 & -0.083801 \\ -0.000138 & 0.007979 & -0.052927 & 0.062607 \end{bmatrix}
$$

was examined. If l_2-scaling constraints were satisfied by choosing $\overline{\Delta}_k$ and $\hat{\Delta}_l$ appropriately [20], the filter coefficients in (2) were derived in case $\overline{\gamma}_k = \hat{\gamma}_l = 1$ for $k, l = 1, 2, 3$ as

$$
\begin{aligned}
[\, \alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,] &= [\, 0.832776 \ \ 0.586915 \ \ 0.166560 \,] \\
[\, \beta_1 \ \ \beta_2 \ \ \beta_3 \,] &= [\, 1.084549 \ \ 0.851201 \ \ 0.344142 \,] \\
[\, \overline{\Delta}_1 \ \ \overline{\Delta}_2 \ \ \overline{\Delta}_3 \,] &= [\, 0.992289 \ \ 0.840742 \ \ 0.457913 \,] \\
[\, \hat{\Delta}_1 \ \ \hat{\Delta}_2 \ \ \hat{\Delta}_3 \,] &= [\, 0.663843 \ \ 0.580255 \ \ 0.323990 \,]
\end{aligned}
$$

$$
[\, r_{kl} \,] = 10^{-1} \begin{bmatrix} 0.1942100 & 0.460026 & 0.370760 & 0.246578 \\ 0.6359199 & 1.877514 & 1.103626 & 1.163573 \\ 0.7625064 & 2.417902 & 3.332929 & 1.812459 \\ 0.5181348 & 1.905871 & 2.841787 & 5.255824 \end{bmatrix}
$$

The results obtained by minimizing M_ρ in (9) and $M_{S\rho}$ in (10) with respect to vector $\boldsymbol{\gamma}(\cdot) = [\overline{\gamma}_1, \cdots, \overline{\gamma}_m, \hat{\gamma}_1, \cdots, \hat{\gamma}_n]^T$, subject to l_2-scaling constraints with exhaustive search in a finite element space were summarized in Table I where the optimal vectors $\boldsymbol{\gamma}(M_\rho^{opt})$ obtained here and $\boldsymbol{\gamma}(M_{S\rho}^{opt})$ in [20] can be written as

$$
\begin{aligned}
\boldsymbol{\gamma}(M_\rho^{opt}) &= [\, 0.750 \ \ 0.625 \ \ 0.750 \ \ 0.625 \ \ 1.000 \ \ 1.000 \,]^T \\
\boldsymbol{\gamma}(M_{S\rho}^{opt}) &= [\, -0.875 \ \ 0.750 \ \ 0.750 \ \ 0.500 \ \ 1.000 \ \ 1.000 \,]^T
\end{aligned}
$$

TABLE I. Performance Comparison

	Ref. [7]	Ref. [20]	Proposed
Minimized l_2-Sensitivity	102.0064	104.6422	98.3040
Number of Nontrivial Coefficients	40	30	32

From Table I, it is observed that the proposed technique results in the minimum l_2-sensitivity among them where there are at most $4(m+n) + mn + 1$ nontrivial coefficients. The method in [7] yields a little larger l_2-sensitivity and its filter implementation is costly where $m^2 + mn + n^2 + 2(m+n) + 1$ nontrivial coefficients exist. The filter obtained by [20] is less expensive in terms of the filter implementation cost, but it has larger l_2-sensitivity where at most $4(m+n) + mn - 1$ nontrivial coefficients exist in the state-space realization of (3).

V. Conclusion

A new expression of evaluating l_2-sensitivity for generalized direct-form II structure (described by Eq. (2) or illustrated in Fig. 2) of 2-D separable-denominator digital filters has been derived and analyzed. The l_2-sensitivity measure has been compared with that deduced in a recent study of generalized direct-form II state-space realization (described by Eq. (3)) of 2-D separable-denominator digital filters [20]. In a numerical example, the l_2-sensitivity measure have been minimized with respect to the free parameters subject to l_2-scaling constraints by exhaustive search in a finite element space. Finally, the numerical results have been compared with those obtained by the techniques in [7] and [20].

References

[1] M. N. S. Swamy and P. K. Rajan, "Symmetry in 2-D filters and its application," in *Multidimensional Systems: Techniques and Applications*, S. Tzafestas, Ed. New York: Marcel Dekkar, 1986, ch. 9.

[2] H. C. Reddy, I.-H. Khoo and P. K. Rajan, "Application of symmetry: 2-D polynomials, Fourier transform, and filter design," in *The Circuits and Filters Handbook*, W. K. Chen, Ed., 3rd ed. Boca Raton, FL: CRC, 2009.

[3] H. C. Reddy, I.-H. Khoo and P. K. Rajan, "2-D symmetry: Theory and filter design applications," *IEEE Circuits Syst. Mag.*, vol. 3, no. 3, pp. 4-33, 3rd Q., 2003.

[4] P.-Y. Chen, L.-D. Van, I.-H. Khoo, H. C. Reddy and C.-T. Lin, "Power-efficient and cost-effective 2-D symmetry filter architectures," *IEEE Trans. Circuits Syst.-I*, vol. 58, no. 1, pp. 112-125, Jan. 2011.

[5] T. Hinamoto, T. Takao and M. Muneyasu, "Synthesis of 2-D separable-denominator digital filters with low sensitivity," *J. Franklin Institute*, vol.329, pp.1063-1080, 1992.

[6] T. Hinamoto and Y. Sugie, "L_2-sensitivity analysis and minimization of 2-D separable-denominator state-space digital filters," *IEEE Trans. Signal Process.*, vol. 50, pp. 3107-3114, Dec. 2002.

[7] T. Hinamoto, Y. Shibata and W.-S. Lu, "Minimization of L_2-sensitivity for 2-D separable-denominator state-space digital filters subject to L_2-scaling constraints," in *Proc. Asia Pasific Conf. Circuits Syst. (APCCAS 2006)*, pp. 390-393, Singapore, Dec. 2006.

[8] G. Li and M. Gevers, "Roundoff Noise minimization using delta-operator realizations," *IEEE Trans. Signal Process.*, vol. 41, no. 2, pp. 629-637, Feb. 1993.

[9] K. Premaratne, M. M. Ekanayake, J. I. Suarez and P. H. Bauer, "Two-dimensional delta-operator formulated discrete-time systems: State-space realization and its coefficient sensitivity properties", *IEEE Trans. Signal Process.*, vol. 46, pp. 3445-3449, Dec. 1998.

[10] J. Kauraniemi, T. I. Laakso, I. Hartimo and S. J. Ovaska, "Delta operator realizations of direct-form IIR filters," *IEEE Trans. Circuits Syst.-II*, vol. 45, no. 1, pp. 41-52, Jan. 1998.

[11] N. Wong and T.-S. Ng, "Roundoff noise minimization in a modified direct-form delta operator IIR structure," *IEEE Trans. Circuits Syst.-II*, vol. 47, no. 12, pp. 1533-1536, Dec. 2000.

[12] N. Wong and T.-S. Ng, "A generalized direct-form delta operator-based IIR filter with minimum noise gain and sensitivity," *IEEE Trans. Circuits Syst.-II*, vol. 48, no. 4, pp. 425-431, Apr. 2001.

[13] G. Li and Z. Zhao, "On the generalized DFIIt structure and its state-space realization in digital filter implementation," *IEEE Trans. Circuits Syst.-I*, vol. 51, no. 4, pp. 769-778, Apr. 2004.

[14] I.-H. Khoo, H. C. Reddy and P. K. Rajan, "Symmetry study for delta-operator-based 2-D digital filters," *.IEEE Trans. Circuits Syst.-I*, vol. 53, pp. 2036-2047, Sep. 2006.

[15] I.-H. Khoo, H. C. Reddy and P. K. Rajan, "Efficient design of delta operator based 2-D IIR filters using symmetrical decomposition," in *Proc. 2008 IEEE Int. Symp. Circuits Syst. (ISCAS 2008)*, Seattle, Washington, pp. 700-703, May 2008.

[16] I.-H. Khoo, H. C. Reddy, L.-D. Van and C. T. Lin, "Delta operator based 2-D VLSI filter structures without global broadcast and incorporation of the quadrantal symmetry," in *Proc. 2012 IEEE Int. Symp. Circuits Syst. (ISCAS 2012)*, Seoul, Korea, pp. 3190-3193, May 2012.

[17] C. Huang, H. Xu, Y. Chen, H. Song and J. Zhang, "A new delta operator based IIR lattice filter structure," in *Proc. 2015 IEEE Int. Conf. Digital Signal Process. (DSP 2015)*, Singapore, pp. 44-47, July 2015.

[18] T. Hinamoto, A. Doi and W.-S. Lu, "Generalized direct-form II realization of 2-D separable-denominator digital filters", in *Proc. 2018 IEEE Int. Symp. Circuits Syst. (ISCAS 2018)*, Florence, Italy, May 2018.

[19] T. Hinamoto, A. Doi and W.-S. Lu, "Roundoff noise analysis for generalized direct-form II structure of 2-D separable-denominator digital filters", in *Proc. 2018 26th European Signal Process. Conf. (EUSIPCO 2018)*, pp. 455-459, Roma, Italy, Aug. 2018.

[20] T. Hinamoto, A. Doi and W.-S. Lu, "l_2-sensitivity analysis and minimization for generalized direct-form II realization of 2-D separable-denominator digital filters", in *Proc. 2019 IEEE Int. Symp. Circuits Syst. (ISCAS 2019)*, Sapporo, Japan, May 2019.

[21] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Autom. Control*, vol. AC-20, pp. 1-10, Feb. 1975.

[22] S.-Y. Kung, B. C. Levy, M. Morf and T. Kailath, "New results in 2-D systems theory, Part II: 2-D state-space models—realization and the notions of controllability, observability, and minimality," *Proc. IEEE*, vol. 65, no. 6, pp. 945-961, June 1977.