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BALANCED APPROXIMATION OF TWO-DIMENSIONAL AND DELAY-DIFFERENTIAL SYSTEMS*

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Abstract

A generalized balanced approximation method for reducing two-dimensional (2-D) and delay differential models is given. This is possible because of the form the Gramians of such systems take. In particular it is shown that the reachability and observability Gramians can be expressed as integrals of a certain system related function along the imaginary axis or unit circle of the complex plane. The Gramians provide two sets of system invariants which lead in a natural way to the concept of a generalized balanced realization, which in turn leads to a way to do model reduction. This paper is based on the report [0].

I. Introduction

Approximating a model of a linear dynamical system by a lower order one can considerably simplify various analysis and synthesis procedures. The appearance of the balanced approximation method of model reduction, initiated by Moore [1], has proved to be most significant because of its desirable properties, such as good error bounds, computational simplicity, stability, and its close connection to robust multivariable control [2]-[6].

When one considers infinite-dimensional systems, such as those modeled by two-dimensional (2-D) difference equations or differential-difference equations results on the model reduction question are only starting to appear, see [7]-[9] for example. Moreover, extension of the balanced approximation method to these systems is not available to date.

In this paper, we give a generalized balanced approximation method for reducing 2-D difference and differential-difference models. To this end the concept of a balanced realization is extended to these systems in a natural manner, which is made possible through the use of certain complex integral representations of the Gramians of the systems considered. Some preliminaries are given in the next section. In particular, we show that the reachability and the observability Gramians of a one-dimensional linear time-invariant system can be expressed as complex integrals of certain system-related functions along the imaginary axis or the unit circle, depending on whether the system considered is continuous or discrete-time. In section III, for the Roesser model of a stable 2-D difference (discrete) type system, generalized Gramians are defined in terms of double integrals of similar system-related functions on the unit bi-circle. These Gramians provide two sets of system invariants, leading naturally to the concept of a generalized balanced realization. To obtain a 2-D balancing transformation, the upper-left and the lower-right submatrices of the Gramians should be positive definite. Several results regarding their computation and positive definiteness are given. Based on these observations, a balanced approximation method is proposed. Section IV is devoted to an analogous study for differential-difference (delay-differential) type systems. A key point for reducing 2-D and delay systems using the balanced approximation method is to compute the upper-left and the lower-right parts of their Gramians. In section V a detailed analysis on this issue is carried out using a Lyapunov approach. Numerical experience has shown that the algorithms can provide useful approximate models at least from the (L^∞) frequency domain point of view. Examples are provided in the report [0].

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II. Preliminaries

Consider a minimal realization of a stable discrete-time system as represented by one-dimensional difference equations

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \qquad k = 0,1,\ldots \tag{1}$$

with $x(k) \varepsilon R^n$, $u(k) \varepsilon R^m$, and $y(k) \varepsilon R^p$. Here R denotes the field of real number and $A \varepsilon R^{nxn}$, $B \varepsilon R^{nxm}$, and $C = R^{pxn}$.

Define the reachability and the observability Gramians by

$$K_d \equiv \sum_{i=0}^{\infty} A^i BB^T (A^T)^i \quad , \text{ and } \quad W_d \equiv \sum_{i=0}^{\infty} (A^T)^i C^T C A^i \tag{2}$$

respectively. An asymptotically stable system (i.e. $\det(\lambda I - A) \neq 0$ for $|\lambda| \geq 1$) is said to be balanced whenever

$$K_d = W_d = \Sigma = \mathrm{diag}\{\sigma_1 \ldots \sigma_n\} \ , \ \sigma_1 \geq \ldots \geq \sigma_n > 0. \tag{3}$$

Assume that system (1) has been balanced. We may regard the Gramians as certain measures of input-to-state and state-to-output couplings, and the balanced realization provides the system a coordinate setting where these couplings are equally weighted so that those state components which are weakly coupled may be discarded [10]. Actually, when

$$\sigma_1 \geq \ldots \geq \sigma_r >> \sigma_{r+1} \geq \ldots \geq \sigma_n > 0$$

we partition A, B, and C of (1) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \ , \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \ , \text{ and } C = [C_1 \quad C_2]$$

with $A_{11} \varepsilon R^{rxr}$, $B_1 \varepsilon R^{rxm}$, and $C_1 \varepsilon R^{pxr}$, and thereby obtain a reduced model with system matrices

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(A_{11},B_1,C_1). It turns out [4] that such a lower-order model represents a good approximation of the original one in terms of the L^∞-norm.

In the continuous-time case, let (F,G,H) be system matrices of a minimal realization of an asymptotically stable (i.e. $\det(\lambda I-F)\neq 0$ for real part $\lambda>0$) linear continuous system of dimension n. Define the reachability and the observability Gramians by

$$K_c = \int_0^\infty e^{Ft}GG^Te^{F^Tt}dt \text{ , and } W_c = \int_0^\infty e^{F^Tt}H^THe^{Ft}dt \quad (4)$$

respectively. The balanced realization of (F,G,H) can similarly be defined and the above reduction procedure also applies.

Given a minimal realization of a linear time-invariant system, a nonsingular coordinate transformation in the state space R^n is said to be a balancing transformation if the transformed realization is balanced. As is seen in [1][2], a key step for obtaining a balancing transformation is to compute the Gramians of the system considered. One way of doing so is to solve the following Lyapunov equations

$$AK_dA^T-K_d = -BB^T \text{ ,} \quad (5a)$$

$$A^TW_dA-W_d = -C^TC \text{ ,} \quad (5b)$$

$$FK_c+K_cF^T = -GG^T \text{ , and} \quad (6a)$$

$$F^TW_c+W_cF = -H^TH \text{ .} \quad (6b)$$

An efficient and reliable algorithm for solving (6) can be found in [11]. In the report [0], an algorithm for computing a balancing transformation as given by Laub [2] is listed.

Alternatively these Gramians can also be obtained through certain complex integrals. To do this define

$$f_d(z) = (zI-A)^{-1}B \text{ ,} \quad (7a)$$

$$g_d(z) = C(zI-A)^{-1} \text{ ,} \quad (7b)$$

$$f_c(s) = (sI-F)^{-1}G \text{ , and} \quad (8a)$$

$$g_c(s) = H(sI-F)^{-1} \text{ ,} \quad (8b)$$

and then the Gramians can be found as indicated in the following theorem.

Theorem 1: Let K_d, W_d, K_c and W_c be given as in (2) and (4), then

$$K_d = \frac{1}{2\pi j}\oint_{|z|=1} f_d(z)f_d^*(z)\,\frac{dz}{z} \text{ ,} \quad (9a)$$

$$W_d = \frac{1}{2\pi j}\oint_{|z|=1} g^*_d(z)f_d(z)\,\frac{dz}{z} \text{ ,} \quad (9b)$$

$$K_c = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} f_c(s)f_c^*(s)ds \text{ ,} \quad (10a)$$

and

$$W_c = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} g^*_c(s)g_c(s)ds \text{ .} \quad (10a)$$

Here * denote conjugate transpose.

proof: The proof can be found in the report [0].

It will be seen that the above integral representations of the Gramians make it possible to extend the concept of a 1-D balanced realization to 2-D and delay-differential type systems in a natural way as detailed in the next section.

III. Approximations of 2-D Systems

Consider a 2-D discrete-time system realized as a Roesser model (two-dimensional difference equations)

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j) \equiv Ax + Bu \quad (11a)$$

$$y(i,j) = [C_1 \quad C_2]\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \equiv Cx \quad (11b)$$

with $A_1 \varepsilon R^{n_1 x n_1}$, $A_4 \varepsilon R^{n_2 x n_2}$, $B_1 \varepsilon R^{n_1 x m}$, and $C_1 \varepsilon R^{p x n_1}$. The 2-D z-transform of (11) yields the transfer function

$$Q(z_1,z_2) = C[I(z_1,z_2)-A]^{-1}B \equiv \frac{N(z_1,z_2)}{p(z_1,z_2)} \text{ .} \quad (12)$$

Here

$$I(z_1,z_2) = z_1I_{n_1} \oplus z_2I_{n_2} \text{ ,} \quad (13)$$

$$p(z_1,z_2) = \det[I(z_1,z_2)-A] \text{ ,} \quad (14)$$

and $\oplus$ denotes direct sum. Throughout this section we assume that

$$p(z_1,z_2)\neq 0 \text{ for } (z_1,z_2)\varepsilon\{(z_1,z_2) \mid |z_1|\geq 1,\ |z_2|\geq 1\} \quad (15)$$

to guarantee the BIBO stability of the system.

Similar to (7), let

$$F_2(z_1,z_2) = [I(z_1,z_2)-A]^{-1}B \quad (16a)$$

and

$$G_2(z_1,z_2) = C[I(z_1,z_2)-A]^{-1} \text{ .} \quad (16b)$$

The generalized reachability and observability Gramians of (11) are then defined as

$$K_2 = \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1} F_2(z_1,z_2)F^*{}_2(z_1,z_2)\frac{dz_2}{z_2}\frac{dz_1}{z_1} \quad (17a)$$

and

$$W_2 = \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1} G^*{}_2(z_1,z_2)G_2(z_1,z_2)\frac{dz_2}{z_2}\frac{dz_1}{z_1} \quad (17b)$$

respectively.

Notice that a 1-D discrete-time model can be regarded as a special case of (11) when A_2,A_3,A_4,B_2, and C_2 are taken to be zero matrices with appropriate dimensions. In this case the upper-left submatrices of dimension n_1 of K_2 and W_2 then coincide with K_d and W_d in (9) respectively, indicating that (17) is a natural generalization for the 2-D case. Moreover, just as in the 1-D case, the generalized Gramians provide certain system invariants which play an essential role in extending the concept of balanced realization to the 2-D case. To show this, observe

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first that a 2-D similarity transformation should be of the form $T=T_1 \oplus T_2$ where $T_1 \epsilon R^{n_1 \times n_1}$ and $T_2 \epsilon R^{n_2 \times n_2}$ are nonsingular, and using this transformation leads to an equivalent realization of $Q(z_1,z_2)$ as $(\hat{A},\hat{B},\hat{C}) = (T^{-1}AT, T^{-1}B, CT)$. Next, denote the reachability and the observability Gramians of $(\hat{A},\hat{B},\hat{C})$ by $\hat{K}_2$ and $\hat{W}_2$ respectively and partition K_2, W_2, $\hat{K}_2$, and $\hat{W}_2$ as

$$K_2 = \begin{bmatrix} K_{11} & K_{12} \\ K^T_{12} & K_{22} \end{bmatrix}, \quad W_2 = \begin{bmatrix} W_{11} & W_{12} \\ W^T_{12} & W_{22} \end{bmatrix},$$

$$\hat{K}_2 = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}^T_{12} & \hat{K}_{22} \end{bmatrix}, \text{ and } \hat{W}_2 = \begin{bmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}^T_{12} & \hat{W}_{22} \end{bmatrix}; \quad (18)$$

where K_{11}, W_{11}, $\hat{K}_{11}$, and $\hat{W}_{11}$ are of dimension n_1. Then we have the following result.

Theorem 2: The eigenvalues of $K_{11}W_{11}$ and $K_{22}W_{22}$ are invariant under 2-D similarity transformations.

proof: By (17), we have that

$$\hat{K}_2 = T^{-1}K_2T^{-T} \quad \text{and} \quad \hat{W}_2 = T^TW_2T . \quad (19)$$

So it follows that

$$\hat{K}_{11} = T_1^{-1}K_{11}T_1^{-T} ,$$

$$\hat{K}_{22} = T_2^{-1}K_{22}T_2^{-T} ,$$

$$\hat{W}_{11} = T_1^TW_{11}T_1 ,$$

and

$$\hat{W}_{22} = T_2^TW_{22}T_2 .$$

Therefore $\hat{K}_{11}\hat{W}_{11} = T_1^{-1}K_{11}W_{11}T_1$

and $\hat{K}_{22}\hat{W}_{22} = T_2^{-1}K_{22}W_{22}T_2$. (20)

□

Remark: By (19), $\hat{K}_2\hat{W}_2 = T^{-1}K_2W_2T$ meaning that the eigenvalues of K_2W_2 are also invariant under 2-D similarity transformations. It should be noted, however, that since a 2-D similarity transformation must be block-diagonal, it is in general impossible to find a transformation $T = T_1 \oplus T_2$ such that both $\hat{K}_2$ and $\hat{W}_2$ are diagonal and equal. Theorem 2 combined with the above observation leads to the following definition.

Definition 1: Realization $(\hat{A},\hat{B},\hat{C})$ of a 2-D transfer function is said to be balanced if

$$\hat{K}_{11} = \hat{W}_{11} = \text{diag}(\sigma_{11},\sigma_{12},\ldots,\sigma_{1n_1}) \quad (21a)$$

and

$$\hat{K}_{22} = \hat{W}_{22} = \text{diag}(\sigma_{21},\sigma_{22},\ldots,\sigma_{2n_2}) \quad (21b)$$

with $\sigma_{11} > \sigma_{12} > \ldots > \sigma_{1n_1} > 0$ and $\sigma_{21} > \sigma_{22} > \ldots > \sigma_{2n_2} > 0$.

To obtain a 2-D balancing transformation, K_{ii} and W_{ii} (i=1,2) need to be positive definite. We have the following result regarding the positive definiteness.

Theorem 3: K_{11} and K_{22} are positive definite if (A,B) is locally reachable, namely

$$\text{rank}[M(1,0) \;\; M(0,1) \;\ldots\; M(n_1,n_2)] = n_1 + n_2 . \quad (22)$$

W_{11} and W_{22} are positive definite if (A,C) is locally observable, that is

$$\text{rank} \begin{bmatrix} C \\ CA_{01} \\ \vdots \\ CA_{0n_2} \\ CA_{10} \\ \vdots \\ CA_{1n2} \\ \vdots \\ CA_{n_1,n_2-1} \end{bmatrix} = n_1+n_2 . \quad (23)$$

Here A_{ij} are recursively defined as

$$A_{ij} = A_{10}A_{i-1,j} + A_{01}A_{i,j-1} \text{ for } i \geq 0 , j \geq 0 \quad (24)$$

with

$$A_{00} = I_{n_1+n_2} , \quad A_{10} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix},$$

$$A_{-i,j} = A_{i,-j} = 0 \quad \text{for } i > 0 , j > 0,$$

and

$$M(i,j) = A_{i-1,j} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A_{i,j-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} .$$

proof: The proof can be found in the report [0].

It has become clear that under the conditions of Theorem 3 that a 2-D balancing transformation T can be found by using K_{ii} and W_{ii} (i=1,2) as matrices K and W of Algorithm 1 (see Appendix 1 of the report [0]) to obtain two 1-D balancing transformations T_1 and T_2, and then form $T = T_1 \oplus T_2$.

A crucial step for obtaining such a T is to compute K_{ii} and W_{ii} (i=1,2). This computation will be detailed in section V. At this point it is also of interest to note the following result.

Theorem 4: The Gramians K_2 and W_2 defined in (17) can be computed as

$$K_2 = \sum_{i,j=o}^{\infty} M(i,j)M^T(i,j) \quad (25a)$$

and

$$W_2 = \begin{bmatrix} (\sum_{i,j=o}^{\infty} A^T_{ij}C^TCA_{ij})_{11} & (\sum_{i,j=o}^{\infty} A^T_{ij}C^TCA_{i-1,j+1})_{12} \\ (\sum_{i,j=o}^{\infty} A^T_{ij}C^TCA_{i+1,j-1})_{21} & (\sum_{i,j=o}^{\infty} A^T_{ij}C^TCA_{ij})_{22} \end{bmatrix} \quad (25b)$$

where $(\cdot)_{11}$ denotes the upper-left submatrix of dimension n_1 of the matrix involved, and $(\cdot)_{12}$, $(\cdot)_{21}$, and $(\cdot)_{22}$ are similarly defined.

proof: The proof can be found in the report [0].

Based on these observations, it is now a simple matter to describe a 2-D balanced approximation approach, that is, once a 2-D balanced realization, say $(\hat{A},\hat{B},\hat{C})$, is found, a reduced 2-D model (system matrices) $(\hat{A}_r,\hat{B}_r,\hat{C}_r)$ of order (r_1,r_2) can be obtained by a further subpartitioning of $(\hat{A},\hat{B},\hat{C})$ as

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$$\hat{A} = \begin{array}{c} \\ r_1\{ \\ r_2\{ \end{array}\!\!\begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{r_1} & \overbrace{\quad}^{r_2} \end{array} \\ \left[\begin{array}{c|c} \hat{A}_{1r} & \hat{A}_{2r} \\ \hline \hat{A}_{3r} & \hat{A}_{4r} \end{array}\right] \end{array}, \quad \hat{B} = \begin{array}{c} r_1\{ \\ r_2\{ \end{array}\!\!\left[\begin{array}{c} \hat{B}_{1r} \\ \hline \hat{B}_{2r} \end{array}\right], \tag{26}$$

$$\text{and } \hat{C} = [\overbrace{\hat{C}_{1r}}^{r_1} \mid \overbrace{\hat{C}_{2r}}^{r_2}]$$

and then taking as the approximation the triple

$$\hat{A}_r = \left[\begin{array}{c|c} \hat{A}_{1r} & \hat{A}_{2r} \\ \hline \hat{A}_{3r} & \hat{A}_{4r} \end{array}\right], \ \hat{B}_r = \left[\begin{array}{c} \hat{B}_{1r} \\ \hline \hat{B}_{2r} \end{array}\right], \text{ and } \hat{C}_r = [\hat{C}_{1r} \mid \hat{C}_{2r}]. \tag{27}$$

An example illustrating this reduction can be found in the report [0].

IV Approximation of Delay-Differential Systems

Realization of retarded and neutral delay-differential type systems has been a subject of study for some time. It was demonstrated [13]-[15] that given a transfer function $Q(z,s)$ of a retarded or a neutral delay-differential type system, there exists a neutral type of realization described as

$$\begin{bmatrix} h(t+1) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} h(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \equiv A \begin{bmatrix} h \\ x \end{bmatrix} + Bu \tag{28a}$$

$$y(t) = [C_1 \quad C_2] \begin{bmatrix} h(t) \\ x(t) \end{bmatrix} \equiv C \begin{bmatrix} h \\ x \end{bmatrix} \tag{28b}$$

with $A_1 \varepsilon R^{n_1 x n_1}$, $A_4 \varepsilon R^{n_2 x n_2}$, $B_1 \varepsilon R^{n_1 x m}$, and $C_1 \varepsilon R^{p x n_1}$. Throughout this section, it is assumed that

$$\det[I(z,s) - A] \neq 0 \text{ , for } |z| \geq 1 \text{ and } \operatorname{Re}s \geq 0 \tag{29}$$

where $I(z,s) = zI_{n_1} \oplus sI_{n_2}$. Condition (29) implies stability independent of delay (i.o.d.) of (28) [16].

Now let

$$F_d(z,s) = [I(z,s) - A]^{-1}B \tag{30a}$$

and

$$G_d(z,s) = C[I(z,s) - A]^{-1} . \tag{30b}$$

Then the generalized reachability and observability Gramians of (28) can be defined as

$$K_{dd} = \frac{1}{(2\pi j)^2} \int_{-j\infty}^{+j\infty} \oint_{|z|=1} F_d(z,s)F^*_d(z,s)\frac{dz}{z}\, ds \ , \tag{31a}$$

$$W_{dd} = \frac{1}{(2\pi j)^2} \int_{-j\infty}^{+j\infty} \oint_{|z|=1} G^*_d(z,s)F_d(z,s)\, \frac{dz}{z}\, ds \tag{31b}$$

respectively. Notice that when A_1, A_2, A_3, B_1, and C_1 are zero matrices of appropriate dimensions, K_{dd} and W_{dd} have their nonzero lower-right submatrices of dimension n_2, being the same as those given by (10).

One of the advantages of employing model (28) is that it enables us to reduce the system's order in s and the number of the delay elements used independently. This is particularly appropriate in the present case since (28) is assumed to be stable i.o.d. so that z and s can be treated as two almost independent variables [16][17].

Analogous to the 2-D case, the similarity transformations of (28) should be of the form $T = T_1 \oplus T_2$, and an equivalent realization of (28) under transfromation T is $(\hat{A},\hat{B},\hat{C}) = (T^{-1}AT,\ T^{-1}B,\ CT)$. By (31), the Gramians of the transformed realization can be computed as

$$\hat{K}_{dd} = T^{-1}K_{dd}T^{-T} \text{ and} \tag{32a}$$

$$\hat{W}_{dd} = T^{T}W_{dd}T \quad . \tag{32b}$$

Now partition K_{dd}, W_{dd}, $\hat{K}_{dd}$ and $\hat{W}_{dd}$ as

$$K_{dd} = \begin{bmatrix} K_{11d} & K_{12d} \\ K^T_{12d} & K_{22d} \end{bmatrix}, \ W_{dd} = \begin{bmatrix} W_{11d} & W_{12d} \\ W^T_{12d} & W_{22d} \end{bmatrix},$$

$$\hat{K}_{dd} = \begin{bmatrix} \hat{K}_{11d} & \hat{K}_{12d} \\ \hat{K}^T_{12d} & \hat{K}_{22d} \end{bmatrix},$$

$$\text{and} \quad \hat{W}_{dd} = \begin{bmatrix} \hat{W}_{11d} & \hat{W}_{12d} \\ \hat{W}^T_{12d} & \hat{W}_{22d} \end{bmatrix} .$$

K_{iid}, W_{iid}, $\hat{K}_{iid}$, and $\hat{W}_{iid}$ are of dimension n_i for $i=1,2$. An argument argument analogous to the proof of Theorem 2 leads to the following result.

Theorem 5: The eigenvalues of $K_{11d}W_{11d}$ and $K_{22d}W_{22d}$ are invariant under above similarity transformations.

Balancing can now be based on the following definition and theorem.

Definition 2: A neutral realization of a delay-differential system, say $(\hat{A},\hat{B},\hat{C})$, is called a balanced realization if

$$\hat{K}_{11d} = \hat{W}_{11d} = \operatorname{diag}(\sigma_{11},\sigma_{12},\ldots,\sigma_{1n_1})$$

and

$$\hat{K}_{22d} = \hat{W}_{22d} = \operatorname{diag}(\sigma_{21},\sigma_{22},\ldots,\sigma_{2n_2}) \ ,$$

with $\sigma_{11} \geq \sigma_{12} \geq \ldots \geq \sigma_{1n_1} \geq 0$ and $\sigma_{21} \geq \sigma_{22} \geq \ldots \geq \sigma_{2n_2} \geq 0$.

Concerning the positive definiteness of K_{iid} and W_{iid} (i=1,2), sufficient conditions are provided next.

Theorem 6: K_{iid} (i=1,2) are positive definite if

$$\operatorname{rank}[M(1,0) \quad M(0,1) \ \ldots \ M(n_1,n_2)] = n_1+n_2 \ . \tag{33a}$$

W_{iid} (i=1,2) are positive definite if

$$\operatorname{rank} \begin{bmatrix} C \\ CA_{01} \\ \vdots \\ CA_{0n_2} \\ CA_{10} \\ \vdots \\ CA_{1n2} \\ \vdots \\ CA_{n_1,n_2-1} \end{bmatrix} = n_1+n_2 \ . \tag{33b}$$

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Here $M(i,j)$ and $A_{i,j}$ are defined as for the 2-D case given previously.

proof: A proof can be found in the report [0].

We now conclude that upon the availability of positive definite K_{iid} and W_{iid} (i=1,2), a balancing transformation of (27) can be obtained through the use of Algorithm 1 of the report [0] where K and W are substituted by K_{iid} and W_{iid} (i=1,2) respectively, resulting in two 1-D balancing transformations T_1 and T_2, and then forming $T = T_1 \oplus T_2$. Furthermore, a reduced model of order (r_1, r_2) can be found by the further subpartitioning procedure as described by (26) and (27).

The computation of K_{iid} and W_{iid} will be considered in the next section.

V. Computational Issues

5.1 2-D Systems: Computation of K_{ii} and W_{ii} (i=1,2)

Truncation Method

By Theorem 4, a straightforward way of computing K_{ii} and W_{ii} (i=1,2) is to use the truncated double-sums, that is,

$$K_{11} \approx [I_{n_1} \;\; 0]\tilde{K}_2 \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \tag{34a}$$

$$K_{22} \approx [0 \;\; I_{n_2}]\tilde{K}_2 \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \tag{34b}$$

$$W_{ii} \approx \Big(\sum_{\ell=0}^{L} \sum_{m=0}^{M} A_{\ell m}^T C^T C A_{\ell m} \Big)_{ii} \quad i = 1,2 \tag{35}$$

where

$$\tilde{K}_2 = \sum_{\ell=0}^{L} \sum_{m=0}^{M} \left(A_{\ell-1,m} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A_{\ell,m-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}\right)\left(A_{\ell-1,m} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A_{\ell,m-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}\right)^T \tag{36}$$

and L and M are positive integers. Through the use of recursive formula (24), the computation procedure of the above finite sums can be readily implemented. In general, when L amd M are large enough, (34) and (35) will represent good approximations of K_{ii} and W_{ii} respectively. On the other hand, experience has shown that even for small n_1 and n_2, this approach needs a considerable amount of computational time.

A Lyapunov Approach

It is easy to verify that a 2-D transfer function associated with realization (A,B,C) can be written as

$$\begin{aligned} C[I(z_1,z_2)-A]^{-1}B &= C_2(z_2I-A_4)^{-1}B_2 + [C_1+C_2(z_2I-A_4)^{-1}A_3] \\ &\quad [z_1I-A_1-A_2(z_2I-A_4)^{-1}A_3]^{-1}[B_1+A_2(z_2I-A_4)^{-1}B_2] \\ &\equiv D_1(z_2)+C_1(z_2)(z_1I-A_1(z_2))^{-1}B_1(z_2) \end{aligned} \tag{37a}$$

or

$$\begin{aligned} C[I(z_1,z_2)-A]^{-1}B &= C_1(z_1I-A_1)^{-1}B_1 + [C_2+C_1(z_1I-A_1)^{-1}A_2] \\ &\quad [z_2I-A_4-A_3(z_1I-A_1)^{-1}A_2]^{-1}[B_2+A_3(z_1I-A_1)^{-1}B_1] \\ &\equiv D_2(z_1)+C_2(z_1)(z_2I-A_2(z_1))^{-1}B_2(z_1) \quad . \end{aligned} \tag{37a}$$

Further notice that

$$K_{11}=[I_{n_1} \;\; 0]K \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} = \frac{1}{(2\pi j)^2} \oint_{|z1|=1} \oint_{|z2|=1} ([I_{n_1} \;\; 0]F_2(z_1,z_2)) ([I_{n_1} \;\; 0]F2(z_1,z_2))^* \frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

which, by (37a), implies that

$$K_{11}= \frac{1}{2\pi j} \oint_{|z_2|=1} \frac{dz_2}{z_2} \frac{1}{2\pi j} \oint_{|z_1|=1} (z_1I-A_1(z_2))^{-1}B_1(z_2)B_1^*(z_2) (\bar{z}_1I-A_1^*(z_2))^{-1}\frac{dz_1}{z_1} \quad . \tag{38}$$

Regarding z_2 as a complex parameter, the second integral of (38) is the positive-definite Hermitian solution of the Lyapunov equation

$$A_1(z_2)K_1(z_2)A_1^*(z_2)-K_1(z_2) = -B_1(z_2)B_1^*(z_2) \ . \tag{39}$$

Thus

$$K_{11} = \frac{1}{2\pi j} \oint_{|z_2|=1} K_1(z_2)\frac{dz_2}{z_2} \quad . \tag{40}$$

The integral in (40) can be computed through the use of the residue theorem. Alternatively, one may noticed that $K(z_2)$ can be factorized as [18]

$$K_1(z_2) = \tilde{K}_1(z_2)\tilde{K}_1^*(z_2) \tag{41}$$

where $\tilde{K}_1(z_2)$ is a proper, stable rational matrix. Thus, if $(\tilde{A}_1,\tilde{B}_1,\tilde{C}_1)$ is a minimal realization of $\tilde{K}_1(z_1)$, then (40) becomes

$$\begin{aligned} K_{11} &= \tilde{C}_1\Big(\frac{1}{2\pi j} \oint_{|z_2|=1} (z_2I-\tilde{A}_1)^{-1}\tilde{B}_1\tilde{B}_1^T(\bar{z}_2I-\tilde{A}_1)^{-T} \frac{dz_2}{z_2}\Big)\tilde{C}_1^T \\ &= \tilde{C}_1\tilde{K}_1\tilde{C}_1^T \end{aligned} \tag{42}$$

where $\tilde{K}_1$ is the positive-definite solution of the Luapunov equation

$$\tilde{A}_1\tilde{K}_1\tilde{A}_1^T - \tilde{K}_1 = -\tilde{B}_1\tilde{B}_1^T \ . \tag{43}$$

Similarly, by (37b) we have

$$K_{22}= \frac{1}{2\pi j} \oint_{|z_2|=1} \frac{dz_1}{z_1} \frac{1}{2\pi j} \oint_{|z_1|=1} (z_2I-A_2(z_1))^{-1}B_2(z_1)B_2^*(z_1) (\bar{z}_2I-A_2^*(z_1))^{-1}\frac{dz_2}{z_2} = \frac{1}{2\pi j} \oint_{|z_1|=1} K_2(z_1)\frac{dz_1}{z_1} \tag{44}$$

where $A_2(z_1)$ and $B_2(z_1)$ are defined in (37b) and $K_2(z_1)$ is the positive Hermitian solution of the Lyapunov equation

$$A_2(z_1)K_2(z_1)A_2^*(z_1) - K_2(z_1) = -B_2(z_1)B_2^*(z_1). \tag{45}$$

Analogous to (40), the last integral in (44) can be evaluated by either applying the residue theorem or factorizing $K_2(z_1)$ as $\tilde{K}_2(z_1)\tilde{K}^*_2(z_1)$ with $\tilde{K}_2(z_1)$ proper and stable and then computing K_{22} as

$$K_{22} = \tilde{C}_2\tilde{K}_2\tilde{C}^T_2$$

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where $\tilde{K}_2$ is the positive-definite solution of the Lyapunov equation

$$\tilde{A}_2\tilde{K}_2\tilde{A}_2^T - \tilde{K}_2 = -\tilde{B}_2\tilde{B}_2^T$$

and $(\tilde{A}_2,\tilde{B}_2,\tilde{C}_2)$ is a minimal realization of $K_2(z_1)$.

It is now obvious that W_{ii} (i=1,2) can also be calculated by applying the above Lyapunov approach with certain straightforward modifications.

5.2 Delay-Differential Systems: Computation of K_{iid} and W_{iid} (i=1,2)

First notice that (37a) and (37b) hold for delay systems modeled by (28) provided that variables z_1 and z_2 are replaced by z and s, respectively. Thus,

$$K_{11d} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} ds\, \frac{1}{2\pi j}\oint_{|z|=1} (zI-A_1(s))^{-1}B_1(s)$$

$$B_1^*(s)(\bar{z}I-A_1^*(s))^{-1}\frac{dz}{z}$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} K_1(s)ds \qquad (46)$$

where

$$A_1(s) = A_1+A_2(sI-A_4)^{-1}A_3 \ ,$$

$$B_1(s) = B_1+A_2(sI-A_4)^{-1}B_2 \ ,$$

and $K_1(s)$ is the positive-definite Hermitian solution of the Lyapunov equation

$$A_1(s)K_1(s)A_1^*(s) - K_1(s) = -B_1(s)B_1^*(s) \ . \qquad (47)$$

Observe that

$$K_{22d} = \frac{1}{2\pi j}\oint_{|z|=1}\frac{dz}{z}\,\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (sI-A_2(z))^{-1}B_2(z)$$

$$B_2^*(z)(\bar{s}I-A_2^*(s))^{-1}\, ds$$

$$= \frac{1}{2\pi j}\oint_{|z|=1} K_2(z)\,\frac{dz}{z} \qquad (48)$$

where

$$A_2(z) = A_4 + A_3(zI-A_1)^{-1}A_2$$

$$B_2(z) = B_2 + A_3(zI-A_1)^{-1}A_2$$

and $K_2(z)$ is the positive-definite Hermitian solution of the Lyapunov equation

$$A_2(z)K_2(z) + K_2(z)A_2^*(z) = -B_2(z)B_2^*(z) \qquad (49)$$

As in the 2-D case, the calculation of the last integrals in (46) and (48) can be carried out by applying either the residue theorem or the factorization approach.

Clearly the above method can also be used to compute W_{iid} (i=1,2). Details of the use of these approximation procedures and calculations can be found in the report [0].

References

[0] W.-S. Lu, E. B. Lee, and Q.-T. Zhang, "Balanced approximation of two-dimensional and delay-differential systems", Technical report of Center for Control Sciences and Dynamical Systems, University of Minnesota, 1986.

[1] B. C. Moore, "Singular-value analysis of linear systems", Proceedings 1978 IEEE CDC, 66-73.

[2] A. J. Laub, "Computation of balancing transformation", Proc. 1980 JACC, FA8-E.

[3] L. Pernebo and L. M. Silverman, "Model reduction via balanced state space representations", IEEE Trans. Automat. Contr., vol. AC-27, 382-387, 1982.

[4] K. Glover, "All optimal Hankel-norm approximation of linear multivariable systems and their L^∞ error bounds", Int. J. Contr., vol. 39, 1115-1193, 1984.

[5] K. Glover and D. Limebeer, "Robust multivariable control system design using optimal reduced order plant models", proc. 1983 ACC, San Francisco, CA.

[6] J. C. Doyle, Lecture Notes, 1984 ONR/Honeywell Workshop on Advances in Multivariable Control, Oct. 8-10, 1984, Minneapolis, MN.

[7] P. N. Paraskevopoulos, "On the model reduction of 2-D systems", Proc. IV Polish-English Seminar on Real Time Process Control, 270-282, 1983.

[8] E. B. Lee, W.-S. Lu, and N. E. Wu, "Approximation of 2-D systems", Proc. Johns Hopkins Conf., 1985.

[9] E. I. Jury and K. Premaratne, "Model reduction of 2-D discrete systems", submitted for publication, 1985.

[10] P. Harshavardhana, E. A. Jonckheere, and L. M. Silverman, "Open and closed loop approximation technique -- An overview", Proc. IEEE 1983 Int. Symp. Circuits and Syst., 126-129, 1983.

[11] R. H. Bartels and G. W. Stewart, "Solution of the matrix equation AX+XB=C", Comm. ACM, vol. 15, 820-826, 1972.

[12] R. P. Roesser, "A discrete state-space model for linear image processing", IEEE Trans. Automat. Contr., vol. AC-20, 1-10, 1975.

[13] E. D. Sontag, "On first-order equations for multidimensional filters", IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-26, 480-482, 1978.

[14] B. C. Lèvy, "2-D polynomial and rational matrices, and their applications for the modeling of 2-D dynamical systems", Tech. Rept., No. M735-11 Inform. Syst. Lab., Stanford Univ., 1981.

[15] S. H. Zak, E. B. Lee, and W.-S. Lu, "Realization of 2-D filters and time-delay systems", to appear in IEEE Trans. on Circuits and Systems, 1986.

[16] E. W. Kamen, "Linear system with commensurate time delays: Stability and stabilization independent of delay", IEEE Trans. Automat. Contr., vol. AC-27, 367-375, 1982.

[17] E. W. Kamen, "Correction to 'Linear systems with commensurate time delay: stability and stabilization independent of delay'", IEEE Trans. Automat. Contr., vol. AC-28, 248-249, 1983.

[18] D. C. Youla, "On the factorization of rational matrices", IRE Trans. Information Theory, vol. IT-7, 172-189, 1961.

[19] W.-S. Lu and E. B. Lee, "Stability analysis for 2-D systems", IEEE Trans. Circuits and Syst., vol. CAS-30, 455-461, 1983.

[20] R. D. Driver, Ordinary and Delay Differential Equations, New York: Springer, 1977.

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