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2-D stability test via 1-D stability robustness analysis

W.-S. LU†

A new two-dimensional stability test is proposed, based on the stability robustness analysis of a relevant 1-D system family. A linear algebra type algorithm for numerical implementation of the test is also described. Two examples then follow, to illustrate the use of the algorithm and the feasibility of the proposed test.

1. Introduction

Stability testing plays a key role in many synthesis techniques for two-dimensional (2-D) digital filters (Jury 1978). From a control viewpoint stability analysis of a 2-D model is also of interest, since a variety of distributed systems, such as retarded or neutral time-delay systems, linear multipass processes and systems governed by certain types of partial differential equations, fit quite naturally in the framework of 2-D system theory (Edwards and Owens 1982, Chiasson 1984).

While 2-D stability analysis in a local state-space setting has been an objective of study for years (Lu and Lee 1983, 1985, 1987, Agathoklis *et al.* 1985), much effort is needed to develop stability tests that can be numerically implemented more efficiently. Based on a fundamental lemma presented in the next section, in conjunction with the 1-D stability robustness result of Martin (1987), we propose a new 2-D stability test which appears to be computationally feasible. Roughly speaking, the proposed test provides a way of obtaining a finite set of 1-D time-invariant discrete systems whose stability implies the stability of the 2-D system considered. We also describe an algorithm to numerically realize the test. The algorithm is purely linear algebraic and can therefore be implemented by using any matrix-computation-oriented software such as MATLAB. Examples then follow to illustrate our results.

2. Model and fundamental lemma

Consider a linear shift-invariant 2-D discrete system modelled in Roesser's local state-space form as

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j) \equiv Ax + Bu \qquad (1)$$

$$y(i,j) = [C_1 \quad C_2] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \equiv Cx$$

where $x^h(i,j) \in R^{n_1}$ and $x^v(i,j) \in R^{n_2}$ form the local state at position (i,j). The

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characteristic polynomial of system (1) is given by

$$c(z_1, z_2) = \det \begin{bmatrix} I - z_1 A_1 & -z_1 A_2 \\ -z_2 A_3 & I - z_2 A_4 \end{bmatrix} \tag{2}$$

where I denotes the identity matrix of appropriate dimension. If (1) has no non-essential singularities of the second kind on the unit bicircle $T^2 = \{(z_1, z_2) : |z_1| = |z_2| = 1\}$, then (1) is bounded-input bounded-output (BIBO) stable if and only if (Goodman 1977)

$$c(z_1, z_2) \neq 0 \quad \text{in } \bar{U}^2 = \{(z_1, z_2) : |z_1| \leqslant 1, |z_2| \leqslant 1\} \tag{3}$$

Define a polynomial matrix $A(z)$ by

$$A(z) = A_{10} + zA_{01} \tag{4}$$

where z is a complex variable,

$$A_{10} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{01} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$$

then the characteristic polynomial may be written as

$$c(z_1, z_2) = \det [I - z_1 A(z)] \tag{5}$$

where $z = z_1^{-1} z_2$. When $(z_1, z_2) \in T^2$, clearly, $z \in T$. The following lemma, that characterizes the BIBO stability of (1) in terms of the spectrum of matrix $A(z)$, is of fundamental importance in order to develop a computationally feasible 2-D stability criterion.

Lemma

Assume (1) has no non-essential singularities of the second kind on T^2, then (1) is BIBO stable if and only if

$$\max_{z \in T} \rho(A(z)) < 1 \tag{6}$$

where $T \triangleq \{z : |z| = 1\}$ and $\rho(A(z))$ denotes the spectral radius of $A(z)$.

Proof

Assume (1) is BIBO stable. For any $z \in T$ and $z_1 \in \bar{U} = \{z : |z| \leqslant 1\}$, let $z_2 = zz_1$. Since $z_2 \in \bar{U}$, (3) and (5) imply (6). Conversely, if (6) holds, then for any $(z_1, z_2) \in T^2$ (3) and (6) imply that

$$c(z_1, z_2) = \det [I - z_1 A(z)] = z_1^{n_1 + n_2} \det [z_1^{-1} I - A(z)] \neq 0$$

Moreover, for any $z \in \bar{U}$, (6) implies that

$$c(z, z) = \det [I - zA(1)] \neq 0$$

It then follows from DeCarlo's criterion (DeCarlo *et al.* 1977) that the system is BIBO stable. □

Define a non-negative real function $r(\theta) = \rho(A(\exp(j\theta)))$. Clearly $r(\theta)$ is continuous and even on $[-\pi, \pi]$. The following corollary is therefore an immediate consequence of the Lemma.

Corollary

Under the same assumption of the Lemma, system (1) is BIBO stable if and only if

$$r(\theta) < 1 \quad \forall\, \theta \in [0, \pi] \tag{7}$$

3. 2-D Stability test via 1-D stability robustness analysis

2-D stability verification through the use of the Lemma means 1-D stability verification of $A(z)$ for every z on the unit circle. On the other hand, one may recognize that somewhat differently from those stability tests proposed in Lu and Lee (1983, 1985), the matrix $A(z)$ involved in the Lemma depends on complex parameter z linearly, making it much easier to estimate the euclidean norm of $A(z)$ under variations of parameter z. This, as will be seen shortly, is crucial in developing a feasible 2-D stability test.

Let $0 = \theta_0 < \theta_1 < \ldots < \theta_L = \pi$ be $L+1$ points that equally partition the interval $[0, \pi]$ and let $z_l = \exp(j\theta_l)$, $0 \leqslant l \leqslant L$. From the Lemma, the 2-D BIBO stability of system (1) implies that for any integer $L > 0$, every $A(z_l)$ is 1-D stable. Conversely, with L large enough, one may expect that the stability of the 1-D system family defined by $\mathfrak{F}_L \equiv \{A(z_l), l = 0, 1, \ldots, L\}$ will imply the 2-D stability of (1) (a 1-D system family is said to be stable if every member in the family is stable). If this is the case, one may then ask how large the family $\mathfrak{F}_L$ should be to guarantee the stability of (1) and how to estimate such an L in terms of the system's parameters. In the rest of this section these questions will be answered via a 1-D stability robustness analysis, leading to a new 2-D stability test.

Assume for each fixed l with $0 \leqslant l \leqslant L$, $A(z_l)$ is stable. For any $\theta \in [\theta_l - \Delta\theta, \theta_l + \Delta\theta]$ where $\Delta\theta = \pi/2L$, we write $A(\exp(j\theta))$ as

$$A(z) = A(z_l) + \Delta A_l \tag{8}$$

where

$$z = \exp(j\theta), \quad \theta \in [\theta_l - \Delta\theta, \theta_l + \Delta\theta], \quad \Delta A_l = (z - z_l)A_{01}$$

and estimate

$$\|\Delta A_l\| = |z - z_l|\, \|A_{01}\| \leqslant \Delta\theta \|A_{01}\| \tag{9}$$

where $\|\Delta A_l\|$ denotes the euclidean norm of ΔA_l. It follows from (9) and Martin (1987, Theorem 5.1 *b*) that matrix $A(z)$ given by (8) will remain stable if

$$\Delta\theta < \frac{\min\limits_{0 \leqslant \omega \leqslant 2\pi} \boldsymbol{\sigma}[\exp(j\omega)I - A(z_l)]}{\|A_{01}\|} \tag{10}$$

where $\boldsymbol{\sigma}(\cdot)$ denotes the smallest singular value of the matrix involved. Let

$$\boldsymbol{\sigma}_l(\omega) = \boldsymbol{\sigma}[\exp(j\omega)I - A(z_l)] \quad \text{and} \quad m_l = \min_{0 \leqslant \omega \leqslant 2\pi} \boldsymbol{\sigma}_l(\omega), \quad 0 \leqslant l \leqslant L \tag{11}$$

then $m_l > 0$ since $A(z_l)$ is assumed to be stable. We further conclude that the 2-D system is BIBO stable if the 1-D system family $\mathfrak{F}_L$ is stable with L satisfying

$$L > \frac{\pi \|A_{01}\|}{2 \min\limits_{0 \leqslant l \leqslant L} m_l} \tag{12}$$

Notice that if system (1) is stable, then such an L always exists. In fact, the stability of (1) implies that

$$m(\theta) \equiv \min_{0 \leqslant \omega \leqslant 2\pi} \boldsymbol{\sigma}[\exp(j\omega)I - A(\exp(j\theta))]$$

as a function of θ on $[0, \pi]$ is continuous and positive. Consequently the minimum of $m(\theta)$ on $[0, \pi]$, denoted by $\mathbf{m}$, is positive. Since for any L, $\min_{0 \leqslant l \leqslant L} m_l \geqslant \mathbf{m}$, (12) is satisfied provided that $L > \bar{L}$, where $\bar{L}$ is defined by

$$\bar{L} = \frac{\pi \| A_{01} \|}{2\mathbf{m}} \tag{13}$$

To numerically find L satisfying (12), m_l defined by (11) must be computed or estimated. Note that with a fixed z_l, $\boldsymbol{\sigma}_l(\omega)$ is a non-negative continuous function of ω on $[0, 2\pi]$. Moreover, if $0 = \omega_0 < \omega_1 < \ldots < \omega_K = 2\pi$ are $K+1$ points that equally partition the interval $[0, 2\pi]$, and $\Delta\omega = \pi/K$, it then follows from a property of singular value decomposition (Golub and Van Loan 1983, p. 286) that for any ω with $|\omega_k - \omega| \leqslant \Delta\omega$

$$|\boldsymbol{\sigma}_l(\omega) - \boldsymbol{\sigma}_l(\omega_k)| \leqslant \|\exp(j\omega) - \exp(j\omega_k)\| \leqslant \Delta\omega, \quad 0 \leqslant k \leqslant K$$

which, along with (11), implies that

$$m_l - \frac{\pi}{K} \leqslant \min_{0 \leqslant k \leqslant K} \boldsymbol{\sigma}_l(\omega_k) \leqslant m_l + \frac{\pi}{K}$$

which may further lead to

$$\min_{0 \leqslant l \leqslant L} m_l - \frac{\pi}{K} \leqslant \min_{0 \leqslant l \leqslant L} \min_{0 \leqslant k \leqslant K} \boldsymbol{\sigma}_l(\omega_k) \leqslant \min_{0 \leqslant l \leqslant L} m_l + \frac{\pi}{K} \tag{14}$$

Denote

$$m_{LK} = \min_{0 \leqslant l \leqslant L} \min_{0 \leqslant k \leqslant K} \boldsymbol{\sigma}_l(\omega_k) \tag{15}$$

it then follows from (14) that

$$\lim_{K \to \infty} m_{LK} = \min_{0 \leqslant l \leqslant L} m_l \tag{16}$$

Note that when K is large enough, making $m_{LK} - \pi/K$ positive, (14) also provides a computable lower bound for $\min_{0 \leqslant l \leqslant L} m_l$ as

$$\min_{0 \leqslant l \leqslant L} m_l \geqslant m_{LK} - \frac{\pi}{K} \tag{17}$$

Thus, from (12) and (17) it is seen that system (1) is BIBO stable if the 1-D family $\mathfrak{F}_L$ is stable, with L satisfying

$$L > \frac{\|A_{01}\|}{\frac{2}{\pi}\left(m_{LK} - \frac{\pi}{K}\right)} \tag{18}$$

where K is chosen such that

$$m_{LK} > \frac{\pi}{K} \tag{19}$$

We now summarize the above analysis as a theorem.

Theorem

Assume (1) has no non-essential singularities of the second kind on T^2, then (1) is BIBO stable if and only if there exist positive integers L and K such that the 1-D system family $\mathfrak{F}_L$ is stable and both (18) and (19) hold.

4. Numerical implementation of the test

The algorithm given below numerically realizes the stability test presented in Theorem 1. Since the test is purely linear algebraic in nature, one may use any reliable matrix computation software such as MATLAB to implement the algorithm.

Algorithm

Step 0. Set $p = 1, q = 1$;

Step 1. Set $L = 2^p$ and verify the stability of $\mathfrak{F}_L$ by computing the spectral radius of $A(z_l)$ for $l = 0, 1, \dots, L-1$. If $\max_{0 \leqslant l \leqslant L-1} \rho(A(z_l)) \geqslant 1$, then the 2-D system considered is unstable, otherwise go to the next step;

Step 2. Set $K = 2^q$ and compute m_{LK} defined by (15);

Step 3. If (19) holds, then go to the next step, otherwise replace q by $q+1$ and go to Step 2.

Step 4. If (18) holds, then the system is BIBO stable, otherwise replace p by $p+1$ and go to Step 1.

Remark

When the 2-D system under verification is unstable, then generically $\rho(A(z)) > 1$, and the above algorithm works well. On the other hand, a careful examination of the algorithm shows that if the system under verification is unstable with $\rho(A(z)) = 1$ then an execution of the algorithm may or may not stop. To avoid this situation one may modify Step 1 of the algorithm by adding: ... If $\max_{0 \leqslant l \leqslant L-1} \rho(A(z_l)) \geqslant 1 - \varepsilon$, then the 2-D system considered is practically unstable, otherwise .... (ε is an acceptable stability margin.)

To illustrate the above algorithm, consider a 2-D system whose system matrix is given by

$$A = \left[\begin{array}{cc|cc} -0\cdot5 & 0\cdot75 & 0\cdot3895 & 0\cdot3895 \\ 0 & 0 & 0 & 0 \\ \hline 0\cdot1423 & 0 & -0\cdot4347 & 0\cdot0981 \\ -0\cdot0342 & 0 & -0\cdot1094 & -0\cdot0903 \end{array}\right]$$

With $K = L = 32$, it is found that $m_{32,32} = 0{\cdot}24$ and $\|A_{01}\| = 0{\cdot}4738$, which implies (18) and (19). Moreover $\mathfrak{F}_{32}$ is proven to be a stable 1-D family. It therefore follows that the 2-D system is BIBO stable. Using software MATLAB on a SUN 3/280 computer, it takes about 40 seconds to complete the stability test for the above system.

In a recent paper by Jury (1988), a modified stability table for 2-D digital filters is proposed. The method verifies the positivity of the Schur–Cohn matrix by using a modified Marden Table. Since Jury's stability test requires less equation solving type of computation, it will in general require less computation time provided that an appropriate computer program is available. This will especially be the case when the system considered has a fairly small stability margin. On the other hand, when the system considered is unstable, our Algorithm may be accomplished at Step 1 with a small integer p, providing a test result quicker than many other stability tests. As an example, consider a 2-D system with the system matrix given below

$$A = \left[\begin{array}{cc|cc} 1{\cdot}5 & 1 & 0 & -0{\cdot}3 \\ -0{\cdot}6 & 0 & 0 & 0{\cdot}11 \\ \hline -1 & 0 & 1{\cdot}2 & -0{\cdot}3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The characteristic equation of the system is given by

$$\begin{aligned} c(z_1, z_2) &= \det \begin{bmatrix} I - z_1 A_1 & -z_1 A_2 \\ -z_2 A_3 & I - z_2 A_4 \end{bmatrix} \\ &= 1 - 1{\cdot}2 z_2 + 0{\cdot}3 z_2^2 - 1{\cdot}5 z_1 + 1{\cdot}8 z_1 z_2 - 0{\cdot}75 z_1 z_2^2 \\ &\quad + 0{\cdot}6 z_1^2 - 0{\cdot}72 z_1^2 z_2 + 0{\cdot}29 z_1^2 z_2^2 \end{aligned}$$

and its stability has been studied by Karan and Srivastava (1986) and Jury (1988) Using the proposed Algorithm, the stability test is accomplished at Step 1 with $p = 1$. Since

$$\max_{0 \leqslant l \leqslant 1} \rho(A(z_l)) = 1{\cdot}4333 > 1$$

we claim that the 2-D system is unstable.

5. Concluding remarks

A 2-D stability test via stability robustness of a relevant 1-D system family along with a purely linear algebraic algorithm has been proposed.

Given a local state-space representation of a 2-D discrete system, we see that it is the spectral property of $A(z)$ that governs the system's BIBO stability. Alternatively, one may begin stability verification with linear polynomial matrix $\hat{A}(z)$ defined by

$$\hat{A}(z) = zA_{10} + A_{01} \tag{20}$$

It is readily shown that with the replacement of $A(z)$ by $\hat{A}(z)$ and with modifications of (15), (18) and (19) as

$$\hat{m}_{LK} = \min_{0 \leqslant l \leqslant L} \min_{0 \leqslant k \leqslant K} \sigma[\exp(j\omega k)I - \hat{A}(z_l)] \tag{21}$$

$$L > \frac{\|A_{10}\|}{\dfrac{2}{\pi}\left(\hat{m}_{LK} - \dfrac{\pi}{K}\right)} \tag{22}$$

and

$$\hat{m}_{LK} > \frac{\pi}{K} \tag{23}$$

respectively, Theorem 1 remains valid.

The benefit that one may gain through this alternative is that if $\|A_{10}\| < \|A_{01}\|$ and $\hat{m}_{LK} \geqslant m_{LK}$, then the number defined by the right-hand side of (22) is smaller than its counterpart in (18), yielding less execution time of the Algorithm.

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