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Brief Paper

A Staircase Model for Unknown Multivariable Systems and Design of Regulators*

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Key Words—Models; robustness; regulator theory; stability, step response.

Abstract—Based on a very simple model, a discrete-time integrating regulator can be designed for a linear time-invariant system. The linear system is not restricted to be scalar with a monotone step response, as required in previously published literature. The number of tuning parameters in the controller depends on the number of steps in the staircase model which can be properly chosen so that the closed-loop system is an asymptotically stable system with a specified tracking speed.

1. *Introduction*

IN THE process control field, since complex processes can be controlled surprisingly well by simple regulators (Åström, 1980a; Richalet and co-workers, 1978), it is convenient and useful to approximate a real but unknown system by a simple model. In this regard, Åström (1980b) has proposed a new design method for any unknown single input–single output (SISO) stable linear time-invariant system with a monotone step response using a model of the form $y(t) = bu(t - T)$, $b > 0$. Let P_M be such an unknown plant and $H(t)$ be the step response of P_M. Åström (1980b) has shown that if

$$2H(T) > H(\infty) \tag{1}$$

and

$$2b > H(\infty) \tag{2}$$

then

$$u_k = (y_k^0 - y_k)/b + u_{k-1} \tag{3}$$

is the robust sampled control law for P_M with a sampling period T, where y_k^0 is the desired output at time kT, $u_k = u(kT)$, $y_k = y(kT)$. When the control law (1) is applied, the closed-loop system was proved to be asymptotically stable.

Under the constraint of condition (1), however, the sampling period might be quite long for some plants resulting in poor controlled response. The assumption that the plant has a monotone step response is restrictive. Also there are no corresponding results for the multivariable case. Moreover, the information about the performance of such a sampled control system is not available even though this is quite important for design engineers.

In this paper, we extend Åström's approach and propose another model, the staircase model, which makes it possible to treat stable but unknown multivariable systems. We show that for such a linear time-invariant plant and a suitable sampling period one can choose a set of parameters in a simple control law derived from a staircase model of the plant so that the asymptotic stability of the resulting closed-loop system is guaranteed. Furthermore, a simple design procedure is given for achieving a required tracking speed.

In Section 2, the staircase model is established for a stable SISO system and then a sampled regulator is derived. Applying this to the unknown plant, the asymptotic stability of the resulting closed-loop system is proved. We also discuss the choice of the regulator parameters so that a required response speed can be achieved. Similar results are given in Section 3 for multivariable systems. Section 4 contains two illustrative examples which indicate that the choice of the parameters in our regulators is easy even for the multivariable case and the closed-loop systems work very well.

2. *Staircase model and robust sampled regulator*

Consider a stable SISO linear time-invariant plant P whose step response $H(t)$ is shown in Fig. 1. We note that the stability of P implies that there exist constants K_1 and β_1 such that

$$|H(t) - H(\infty)| \leqslant K_1 e^{-\beta_1 t}, \quad \text{for } t \geqslant 0. \tag{4}$$

It should be noted that for an unknown system the values K_1 and β_1 in (4) can be estimated by measuring its step response. Let T be the sampling period and $I(t)$ be the unit step function. Then, the function $\bar{H}(t)$ shown in Fig. 1 can be described by

$$\bar{H}(t) = \sum_{j=1}^{N} \delta_j I(t - jT) \tag{5}$$

where integer N and the real sequence $\{\delta_i, 1 \leqslant i \leqslant N\}$ may be adjusted if necessary.

We now consider a model M for the unknown plant P with $\bar{H}(t)$ defined in (5) as its unit step response. Denoting the input and the output of M by $u(t)$ and $y^M(t)$, respectively, the mathematical model for M is

$$y^M(t) = \sum_{j=1}^{N} \delta_j u(t - jT). \tag{6}$$

It is observed that the model

$$y^M(t) = bu(t - T)$$

suggested by Åström (1980b) is a special case of our model if we set $N = 1$ and $\delta_1 = b$.

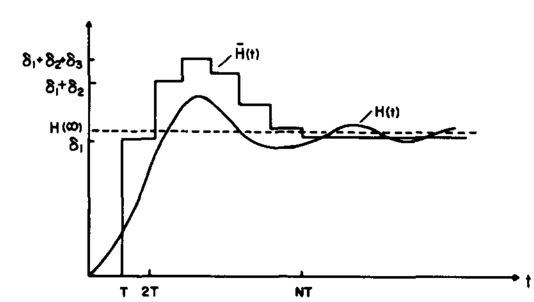

FIG. 1. Step response of P and its staircase model.

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Define $y_k^M = y^M(kT)$, $u_k = u(kT)$, $H_k = H(kT)$, $H_\infty = H(\infty)$ and $u_j = 0$ for all $j < 0$. By (6) we have

$$y_{k+1}^M - y_k^M = \delta_1 u_k - \sum_{j=1}^{N} (\delta_j - \delta_{j+1})u_{k-j}$$

where we have set $\delta_{N+1} = 0$. If the output of M at $(k+1)T$ is desired to be the reference value y_{k+1}^0 and assuming $\delta_1 \neq 0$, the control u_k is

$$u_k = \frac{1}{\delta_1}(y_{k+1}^0 - y_k^M) + \frac{1}{\delta_1}\sum_{j=1}^{N} (\delta_j - \delta_{j+1})u_{k-j}.$$

Applying this to the plant P, the sampled control law becomes

$$u_k = \frac{1}{\delta_1}(y_{k+1}^0 - y_k) + \frac{1}{\delta_1}\sum_{j=1}^{N} (\delta_j - \delta_{j+1})u_{k-j} \qquad (7)$$

which together with a zero-order hold yields the control function

$$u(t) = \begin{cases} u_k & \text{if } t \in [kT, (k+1)T) \\ 0 & \text{if } t < 0. \end{cases} \qquad (8)$$

Theorem 2.1. For the plant P described above and $H_\infty > 0$, the closed-loop system composed of P and regulator (7) will be asymptotically stable, if the parameters N and δ_j $(1 \leqslant j \leqslant N)$ and the sampling period T are chosen so that the conditions given below

$$H_j - H_{j-1} \leqslant \delta_j - \delta_{j+1}, \quad 1 \leqslant j \leqslant N \qquad (9)$$

and

$$H_N > \gamma_1 e^{-\beta_1 NT} \qquad (10)$$

where

$$\gamma_1 = K_1 \frac{1 + e^{-\beta_1 T}}{1 - e^{-\beta_1 T}} \qquad (11)$$

are satisfied.

Proof. Let $h(t)$ be the impulse response of P. Then

$$y_k = \int_0^\infty h(\tau)u(kT - \tau)\,d\tau = \sum_{j=1}^{\infty} (H_j - H_{j-1})u_{k-j}$$

which with (7) yields

$$u_k + \frac{1}{\delta_1}\sum_{j=1}^{N} (H_j - H_{j-1} - \delta_j + \delta_{j+1})u_{k-j} + \frac{1}{\delta_1}\sum_{j=N+1}^{\infty} (H_j - H_{j-1})u_{k-j} = \frac{y_{k+1}^0}{\delta_1} \qquad (12)$$

which must be satisfied by the control sequence $\{u_k\}$. Denoting (12) by

$$\sum_{j=0}^{\infty} a_j u_{k-j} = \frac{y_{k+1}^0}{\delta_1}$$

where $a_0 = 1$, it is observed that if the function

$$A(z) = \sum_{j=0}^{\infty} a_j z^{-j}$$

has the property that $\inf_{|z|\geqslant 1}|A(z)| > 0$, then the related closed-loop system will be asymptotically stable (Desoer and Vidyasagar, 1975).

Note that

$$\inf_{|z|\geqslant 1} |A(z)| \geqslant \inf_{|z|\geqslant 1} \left(1 - \sum_{j=1}^{\infty} |a_j||z|^{-j}\right) = 1 - \sum_{j=1}^{\infty} |a_j|.$$

Consequently, one concludes that $\sum_{j=1}^{\infty} |a_j| < 1$ will guarantee the asymptotic stability of the closed-loop system.

By (9)

$$\sum_{j=1}^{N} |a_j| = \frac{1}{\delta_1}\sum_{j=1}^{N} |H_j - H_{j-1} - \delta_j + \delta_{j+1}| = \frac{1}{\delta_1}(\delta_1 - H_N).$$

By (4) and (11)

$$\sum_{j=N+1}^{\infty} |a_j| = \frac{1}{\delta_1}\sum_{j=N+1}^{\infty} |H_j - H_{j-1}| \leqslant \frac{1}{\delta_1}\sum_{j=N+1}^{\infty} (|H_j - H_\infty| + |H_{j-1} - H_\infty|)$$
$$\leqslant \frac{K_1}{\delta_1}(1 + e^{\beta_1 T}) \sum_{j=N+1}^{\infty} e^{-\beta_1 jT} = \frac{\gamma_1}{\delta_1} e^{-\beta_1 NT}.$$

Hence

$$\sum_{j=1}^{\infty} |a_j| \leqslant \frac{1}{\delta_1}(\delta_1 - H_N + \gamma_1 e^{-\beta_1 NT})$$

which with (10) implies $\sum_{j=1}^{\infty} |a_j| < 1$. This completes the proof.

Remarks.

1. The proposed scheme (7) may be interpreted as an integrating regulator with a compensation for past inputs similar to the one used in a Smith predictor. By (9) we have $\delta_1 \geqslant H_n$ which with (10) implies $\delta_1 > 0$ so that the control law (7) does make sense.

2. The assyumption $H_\infty > 0$ means that there exists an integer N such that (10) holds, since the right side of (10) goes to zero as N goes to infinity. However, even though no explicit restriction should be placed on the sampling period T, it should not be selected to be too small. Otherwise, by (10) and (11), integer N in (10) would be quite large so that too many terms would be involved in the control law (7). It is observed that the regultor gain is inversely proportional to the size of the first step of the staircase which will increase drastically by decreasing the sampling period.

Moreover, it should be noted that the assumption $H_\infty > 0$ is not essential. In fact, for a system with $H_\infty < 0$, Theorem 2.1 remains valid if one replaces (9) and (10) by

$$H_j - H_{j-1} \geqslant \delta_j - \delta_{j+1}, \quad 1 \leqslant j < N \qquad (9')$$

and

$$-H_N > \gamma_1 e^{-\beta_1 NT} \qquad (10')$$

respectively. In this case, one easily concludes that $\delta_1 < 0$ so that (7) does make sense even here.

3. For the purpose of controller design, one should first specify a sampling period T and determine constants K_1 and β_1 in (4) according to the step response data from the plant. The number N can then be chosen by (10). Finally, beginning with δ_N, the sequence $\{\delta_j\}$ can be decided by (9) recursively.

A special case arises when step response H(t) is monotone in which case

$$\sum_1^{\infty} |a_j| = \frac{1}{\delta_1}(\delta_1 - 2H_N - H_\infty)$$

which implies that (9) and

$$H_N > \frac{H_\infty}{2} \qquad (13)$$

guarantee $\sum_1^{\infty} |a_j| < 1$. This leads to the following corollary.

Corollary 2.1. For a stable SISO linear plant with monotone step response, the closed-loop system with regulator (7) will be asymptotically stable if the parameters in (7) are chosen so that (9) and (13) are satisfied.

It is interesting to consider the possibility of choosing the regulator in such a way that the output of the closed-loop system will follow a specified reference value with a desired speed. For

SISO case, described by Desoer and Vidyasagar (1975) we know that $A(z) = 0$ is equivalent to the characteristic equation of the closed-loop system so that requiring the system to have an exponential decay type of error, i.e. $e(t) = y(t) - y^0 \sim e^{-\mu t}$ for some $\mu > 0$ where y^0 is a constant reference value, means that one requires

$$\inf_{|z| \geq e^{-\mu T}} |A(z)| > 0.$$

Note that

$$\inf_{|z| \geq e^{-\mu T}} |A(z)| \geq \inf_{|z| \geq e^{-\mu T}} \left(1 - \sum_{j=1}^{\infty} |a_j||z|^{-j}\right) = 1 - \sum_{j=1}^{\infty} |a_j| e^{j\mu T}.$$

So if

$$\delta_j = \delta_{j+1} + H_j - H_{j-1}, \quad \text{for } 1 \leq j \leq N \tag{14}$$

a straightforward calculation will give

$$\sum_{j=1}^{\infty} |a_j| e^{j\mu T} \leq \gamma_4 \frac{e^{-(N+1)(\beta_1 - \mu)T}}{H_\infty - K_1 e^{-N\beta_1 T}} \tag{15}$$

where $\gamma_4 = K_1(1 + e^{\beta_1 T})/1 - e^{-(\beta_1 - \mu)T}$ and $\mu < \beta_1$ has been assumed.

Theorem 2.2. A stable SISO linear plant P with regulator (7) will have a desired tracking error $ce^{-\mu t}$ for a constant reference value where $0 < \mu < \beta_1$, if the regulator parameters N and $\{\delta_j, 1 \leq j \leq N\}$ satisfy (14) and

$$H_\infty e^{(N+1)(\beta_1 - \mu)T} - K_1 e^{[\beta_1 - \mu(N+1)]T} > \gamma_4 \tag{16}$$

Proof. Equations (14) and (16) imply

$$\sum_{j=1}^{\infty} |a_j| e^{j\mu T} < 1$$

so that

$$\inf_{|z| \geq e^{-\mu T}} |A(z)| > 0$$

i.e. there are no characteristic roots of the closed-loop system in $|z| \geq e^{-\mu T}$.

In view of (16), one can require a better decay rate which is closer to β_1 at the expense of a larger N which means an increase in the number of terms in the regulator (7). Thus, the designer should compromise between the response speed and the complexity of the required regulator.

3. *MIMO case*

Let $G(s) = (g_{ij}(s))$ be a $p \times m$ transfer matrix of a stable MIMO linear time-invariant plant P. Denote the m-dimensional input and the p-dimensional output by $U(t)$ and $Y(t)$, respectively.

Similar to what we did in the previous section, a staircase model can be established as

$$Y^M(t) = P(z^{-1})U(t)$$

where $Y^M(t)$ is the output of the model, z^{-1} is the backward shift operator and

$$P(z^{-1}) = P_1 z^{-1} + \ldots + P_s z^{-s} \tag{17}$$

for some positive integer s and $p \times m$ constant matrices P_k $(1 \leq k \leq s)$. For simplicity, from now on we assume that $p = m$ and $\det P_1 \neq 0$. Thus, a similar derivation will lead to the following regulator

$$U_k = P_1^{-1}(Y_{k+1}^0 - Y_k) + P_1^{-1} \sum_{i=1}^{s} (P_i - P_{i+1})U_{k-i} \tag{18}$$

where $P_{s+1} = 0$, $U_k = U(kT)$, $Y_k = Y(kT)$ and $Y_{k+1}^0 = Y^0[(k+1)T]$ which is the reference value at time $(k+1)T$. The output of regulator (18) and a zero-order hold thus generates the control function $U(t)$ as follows:

$$U(t) = \begin{cases} U_k & \text{if } t \in [kT, (k+1)T) \\ 0 & \text{if } t < 0. \end{cases} \tag{19}$$

Let $H(t) = (H^{ij}(t))$ be the step response matrix whose entry $H^{ij}(t)$ is the step response of the subsystem with $u_j(t)$ and $y_i(t)$ as its input and output respectively where u_j and y_i are the jth and ith components of $U(t)$ and $Y(t)$, respectively. By the stability of the plant P, there exist constants K_2 and β_2 such that

$$\|H(t) - H(\infty)\| \leq K_2 e^{-\beta_2 t} \tag{20}$$

where $\|\cdot\|$ could be any kind of consistent norm on $R^{m \times m}$. In addition, a straightforward calculation gives

$$Y_k = D(z^{-1})U_k \tag{21}$$

when control (19) is applied to the plant P, where

$$D(z^{-1}) = \sum_{i=1}^{\infty} (H_i - H_{i-1})z^{-i} \quad \text{and} \quad H_i = H(iT).$$

Theorem 3.1. For a stable MIMO linear plant P, the closed-loop system with regulator (18) is asymptotically stable if the sampling period T and regulator parameters s and P_r $(1 \leq r \leq s)$ are chosen properly so that the following conditions are satisfied:

(i) H_s^{-1} exists and $\|e^{-\tau s}H_s^{-1}\|$ is bounded uniformly in s for some $0 \leq \tau \leq \beta_2 T$;

(ii) $P_i = P_{i+1} + H_i - H_{i-1}, \quad 1 \leq i \leq s;$ (22)

(iii) $\gamma_2 e^{-s\beta_2 T}\|H_s^{-1}\| < 1;$ (23)

where $\gamma_2 = K_2(1 + e^{-\beta_2 T}/1 - e^{-\beta_2 T})$, are satisfied. The proof of this theorem is similar to that of Theorem 2.1 and is omitted here.

To design such a regulator, the following steps may be followed:

Step 1. Record the step response $H(t)$ for plant P.

Step 2. Choose constants K_2 and β_2 so that (20) holds.

Step 3. For some chosen sampling period T, determine s so that (23) holds. Note that because of condition (i), this step can always be done.

Step 4. Compute the coefficient matrices in (18) by means of (22).

By Theorem 3.1, the regulator (18) becomes

$$U_k = H_s^{-1}(Y_{k+1}^0 - Y_k) + H_s^{-1} \sum_{i=1}^{s} (H_i - H_{i-1})U_{k-i}. \tag{24}$$

Concerning the response speed consideration, one can treat the multivariable case in a similar fashion and obtain the following theorem.

Theorem 3.2. A stable MIMO linear plant P with regulator (24) will have a desired tracking error $ce^{-\mu t}$ for some $0 < \mu < \beta_2$ if $\|e^{-\tau_1 s}H_s^{-1}\|$ is bounded uniformly in s for some $0 \leq \tau_1 < (\beta_2 - \mu)T$ and the parameter s in (24) satisfies

$$s > \frac{\ln(\gamma_5 M)}{(\beta_2 - \mu)T - \tau_1} \tag{25}$$

where $\gamma_5 = K_2(e^{\beta_2 T} + 1/e^{(\beta_2 - \mu)T} - 1)$ and $\|e^{-\tau_1 s}H_s^{-1}\| \leq M$.

4. *Illustrative examples*

Example 4.1. Consider a second-order plant $G(s) = 2/s^2 + 2s + 2$ with non-monotone unit step response $H(t) = 1 - e^{-t}(\cos t + \sin t)$. For the plant one may choose $K_1 = 1.5, \beta_1 = 1$. If $T = 0.5$ and $\mu = 0.4$ is required, by (16) it can be found that $N = 9$. Suppose $y^0(t) = 2$, then (14) and (7) give

$$u_k = \frac{1}{1.013}\left(2 - y_k + \sum_{j=1}^{9} d_j u_{k-j}\right) \tag{26}$$

where

$$d_j = H_j - H_{j-1} = e^{-0.5j}[e^{0.5}(\cos 0.5(j-1) + \sin 0.5(j-1)) - (\cos 0.5j + \sin 0.5j)].$$

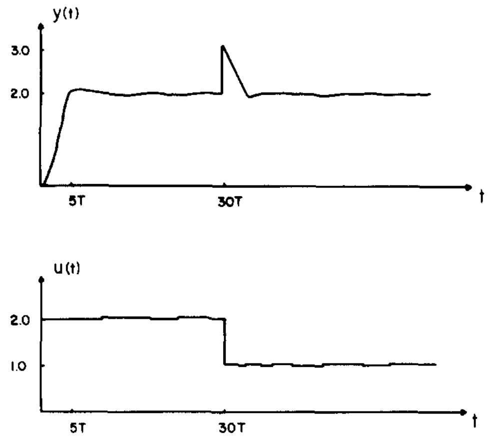

FIG. 2. Example 4.1.

The control u_k and the response of the closed-loop system y_k can be computed by using (26) and $y_k = \sum_{j+1}^{k} d_i u_{k-i}$. Figure 2 shows that the error $y_k^0 - y_k$ decays at a rate no slower than $ce^{-0.4k}$ as required and the output comes back to within $\pm 5\%$ tolerance rapidly after the system was subjected to a unit step disturbance beginning at $t = 30T$.

Example 4.2. Suppose the multivariable plant considered is

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{-2}{s+2} \\ \dfrac{-1}{s+1} & \dfrac{4}{s+2} \end{bmatrix}$$

we now have

$$H(t) = \begin{bmatrix} 1 - e^{-t} & -1 + e^{-2t} \\ -1 + e^{-t} & 2 - 2e^{-2t} \end{bmatrix}$$

and

$$H^{-1}(t) = \begin{bmatrix} \dfrac{2}{1 - e^{-t}} & \dfrac{1}{1 - e^{-t}} \\ \dfrac{1}{1 - e^{-2t}} & \dfrac{1}{1 - e^{-2t}} \end{bmatrix}.$$

One thus can choose $K_2 = 3, \beta_2 = 1$ in (20) and $\tau_1 = 0, M = 3.2$ in (25) which lead to $s = 4$ if we assume $T = 1.0$ and need $\mu = 0.15$. The regulator (24) then is

$$U_k = \begin{bmatrix} 2.037 & 1.019 \\ 1.0 & 1.0 \end{bmatrix} (Y_{k+1}^0 - Y_k) + \begin{bmatrix} 1.749 & 0 \\ 0 & 6.389 \end{bmatrix} \sum_{i=1}^{\infty} \begin{bmatrix} e^{-i} u_{k-i}^1 \\ e^{-2i} u_{k-i}^2 \end{bmatrix}$$

and the output of the system will be

$$Y_k = \begin{bmatrix} 1.172 & -6.389 \\ -3.437 & 12.778 \end{bmatrix} \sum_{i=1}^{k} \begin{bmatrix} e^{-i} u_{k-i}^1 \\ e^{-2i} u_{k-i}^2 \end{bmatrix}.$$

Figure 3 shows that the system designed above works well where the reference vector is $Y_k^0 = [1.5, 2.0]^T$ for $k \in [0, 50T]$ and $Y_k^0 = [-1.5, -2.0]^T$ for $k \in [51T, 100T]$ and the system output is interrupted by a large step disturbance which is $N_k = [2.0, -2.0]^T$ for $k \in [20T, 50T]$ and $N_k = [-2.0, 2.0]^T$ for $k \in [70T, 100T]$.

5. *Conclusions*

It has been shown that reasonably good control can be achieved by establishing a simplified process model and then designing a regulator for it. The results presented in this paper may be developed further along several significant directions. Among them, we would like to mention the attempts of extending current results to the case in which some sort of random noise exists and developing some real-time and recursive algorithm and extending to the case in which the plant is unstable. It might be noted that the model (6) is a special case of the transfer function model $y(t_k) = [B(z)/A(z)]u(t_k)$ for which a general theory is available even in the noisy case. These generalizations are now underway and will be reported in a future paper.

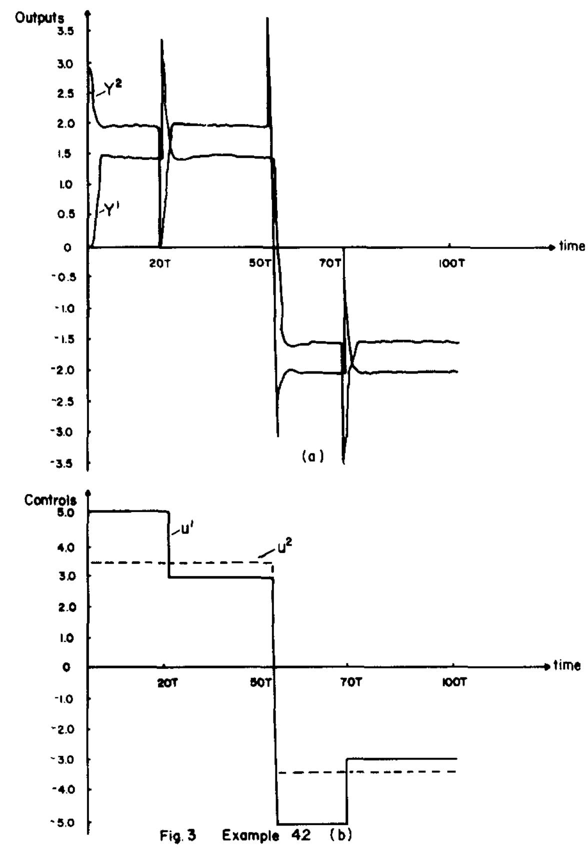

FIG. 3. Example 4.2.

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