$t_1/t_H \sim 361.8$, but we can observe from Fig. 3(b) that if we attempted to approach such a value of gain, the rise time would become prohibitively large.

**Appendix B**

A. Logic Control of the Input Drive Waveform

There is fertile ground for innovation in the development of digital logic circuits to produce near idealized waveforms for the profiled drive signals. It is a particularly attractive proposition to have, even in simplest form, an output sensing system which actually monitors the load conditions rather than a circuit that relies purely on predetermined drive. Such a scheme could take the form as shown in Fig. 4. Here, the initial turn-on pulse is terminated as soon as the collector voltage has gone low (the delay unit to the upper AND gate being equipped with a desired threshold adjustment). Then the transistors are maintained in the on state by the sample-and-hold without the usual over-saturation of the bipolar transistor. The clock-off pulse eventually cancels the sample-and-hold, transmits the turn-off signal, and the latter is cancelled when the collector has gone high (off-state), by the lower AND gate with its inverted input. Alternatively, this AND gate with its output inverter (making NAND) could be reduced, by DeMorgan’s theorem, to an OR gate with complemented inputs.

**References**


**On A Lyapunov Approach to Stability Analysis of 2-D Digital Filters**

W.-S. Lu

**Abstract**—This paper describes an approach to the stability analysis of two-dimensional (2-D) digital filters that are modeled in the Fornasini-Marchesini local state space by

$$x(i+1,j+1) = A_1 x(i,j+1) + A_2 x(i+1,j) + b_1 u(i,j+1) + b_2 u(i+1,j)$$

$$y(i,j) = c x(i,j) + d u(i,j)$$

where $A_1, A_2 \in \mathbb{R}^{n \times n}$. It is known [9] that (1) is asymptotically stable if and only if

$$p(z_1,z_2) \equiv \det (I - z_1 A_1 - z_2 A_2) \neq 0 \text{ for } (z_1, z_2) \notin \tilde{U}$$

where $\tilde{U} = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}$, and I denotes the identity matrix of dimension $n$. In [8] Hinamoto proved the following theorem:

**Theorem I: [8]** (1) is asymptotically stable if there exists a positive definite symmetric $P \in \mathbb{R}^{n \times n}$ such that

$$Q = \begin{bmatrix} \alpha P & 0 \\ 0 & \beta P \end{bmatrix} - A_1^T PA$$

is positive definite where $A = [A_1, A_2]$, and

$$\alpha, \beta > 0, \quad \alpha + \beta = 1$$

A graphic illustration of Theorem 1 was also given in [8] as follows. If filter (1) satisfies (3) and (4), then all zeros of characteristic polynomial $p(z_1,z_2)$ correspond to locations outside the ellipse in the first quadrant shown in Fig. 1, where the ellipse is described by

$$\frac{|z_1|^2}{1/\alpha} + \frac{|z_2|^2}{1/\beta} = 1$$

From this illustration it follows that stable filters with poles (i.e., zeros of $p(z_1,z_2)$) corresponding to the locations in both shaded areas $S_1$ and $S_2$ shown in Fig. 2, for example, do not satisfy (3) and (4). This is to say, no $\alpha$ and $\beta$ can be found to meet (4) such that zeros of $p(z_1,z_2)$ in both areas $S_1$ and $S_2$ are left outside the corresponding ellipse. Thus we see that there is a large number of filters that are stable but their stability cannot be confirmed using Theorem 1. We shall touch upon this matter again in Sec. V using an example.
In the rest of paper we shall consider the Lyapunov equation
\[ Q = \begin{bmatrix} P^{T/2}W_1P^{1/2} & 0 \\ 0 & P^{T/2}W_2P^{1/2} \end{bmatrix} - A^TP^{T/2}RP^{1/2}A \] (8)
where matrices \( P, W_1, W_2 \) and \( R \) are all positive definite. Clearly (8) becomes (3) if \( W_1 = \alpha I, W_2 = \beta I \) and \( R = I \). For this reason we shall call (8) the \textit{generalized} 2-D Lyapunov equation associated with state-space model (1).

### III. MAIN RESULTS

The main theorem, stated and proved below, uses the form of the generalized Lyapunov equation (8) to confirm stability of a 2-D state-space digital filter.

\textbf{Theorem 2:} (1) is asymptotically stable if there are positive definite matrices \( P, W_1, W_2, \) and \( R \) such that matrix \( Q \) given by (8) is positive definite and that
\[ R - W_1 - W_2 \geq 0 \] (9)

\textbf{Proof:} We prove the theorem by contradiction. Suppose the conditions of the theorem are satisfied but (1) is unstable. Then there exists \( (z_1, z_2) \in \mathbb{C}^2 \) such that
\[ \det (I - z_1A_1 - z_2A_2) = 0 \] (10)
Hence there exists \( v \neq 0 \) such that
\[ v = A \begin{bmatrix} z_1I \\ z_2I \end{bmatrix} v \] (11)
where \( A = [A_1 \ A_2] \). We now use (8) and (11) to compute the equation shown at the bottom of the page, where \( v^* \) denotes the complex conjugate transposition of \( v \), \( z^* \) is the complex conjugate of \( z \), and \( v^* \) is the complex conjugate of \( v \). It follows that
\[ v^*P^{T/2}(R - [z_1]^T W_1 - [z_2]^T W_2)P^{1/2}v = -v^*Qv \] (12)
By (10) we note that \( (z_1, z_2) \neq (0, 0) \), hence \( v \neq 0 \). So \( Q > 0 \) means that the right-hand side of (12) is negative. On the other hand, \( [z_1] \leq 1, [z_2] \leq 1 \) and the positive semi-definiteness of \( R - W_1 - W_2 \) imply that \( R - [z_1]^T W_1 - [z_2]^T W_2 \geq 0 \), therefore the left-hand side of (12) is nonnegative, leading to a contradiction. This completes the proof.

Several consequences can be drawn from Theorem 2 immediately.

\textbf{Corollary 2.1:} (1) is asymptotically stable if there is a matrix \( P > 0 \) such that
\[ Q = \begin{bmatrix} P^{T/2}W_1P^{1/2} & 0 \\ 0 & P^{T/2}W_2P^{1/2} \end{bmatrix} - A^TP^{T/2}PA \] (13)
is positive definite, where \( W_1 = U^T \Sigma_1 U, W_2 = U^T \Sigma_2 U \) with \( U \) orthogonal, \( \Sigma_1 = \text{diag} \{ \sigma_1, \ldots, \sigma_n \} \), \( \Sigma_2 = \text{diag} \{ \sigma_1, \ldots, \sigma_2n \} \), \( \sigma_k > 0, \sigma_{2k} > 0 \) and \( \sigma_k + \sigma_{2k} = 1, \quad 1 \leq k \leq n \) (14)

\textbf{Proof:} Let \( R = I \) and note \( I - W_1 - W_2 = 0 \). Applying Theorem 2 completes the proof.

If the orthogonal matrix \( U \) in above corollary is set to be the identity matrix then we obtain the following corollary.

\textbf{Corollary 2.2:} (1) is asymptotically stable if there is a matrix \( P > 0 \) such that
\[ Q = \begin{bmatrix} P^{T/2}W_1P^{1/2} & 0 \\ 0 & P^{T/2}W_2P^{1/2} \end{bmatrix} - A^TP^{T/2}PA \] (15)

\[ v^*P^{T/2}RP^{1/2}v = v^* [z_1I \ z_2I] \begin{bmatrix} P^{T/2}W_1P^{1/2} & 0 \\ 0 & P^{T/2}W_2P^{1/2} \end{bmatrix} [z_1I \ z_2I] v - v^*Qv \]
is positive definite, where \( W_i = \text{diag} \{ \sigma_{i1}, \ldots, \sigma_{i\alpha_i} \} \) (i = 1, 2) satisfy (14).

**Corollary 2.3 (Hinamoto):** (1) is asymptotically stable if there is a matrix \( P > 0 \) such that

\[
Q = \begin{bmatrix}
\alpha P & 0 \\
0 & \beta P
\end{bmatrix} - A^T PA
\]

is positive definite, where

\[
\alpha > 0, \quad \beta > 0 \quad \text{and} \quad \alpha + \beta = 1
\]

**Proof:** This is another special case of Theorem 2 with \( R = I \), \( W_1 = \alpha I \) and \( W_2 = \beta I \). By (17) we have \( R \cdot W_1 \cdot W_2 = 0 \), and the use of Theorem 2 completes the proof.

Note that in all three corollaries matrix \( R \) is assigned to be the identity matrix. As will be seen from the next theorem, this assignment does not lose the generality of Theorem 2.

**Theorem 3:** The sufficient conditions stated in Theorem 2 for (1) to be asymptotically stable and the conditions stated below are equivalent: There exist matrices \( P > 0 \), \( W_1 > 0 \), and \( W_2 > 0 \) such that

\[
Q = \begin{bmatrix}
P^{T/2} W_1 P^{1/2} & 0 \\
0 & P^{T/2} W_2 P^{1/2}
\end{bmatrix} - A^T PA
\]

is positive definite and

\[
I - W_1 - W_2 \succeq 0
\]

**Proof:** Since the stability of a space-state digital filter remains unchanged under a state-variable transformation \( x(i,j) = T \bar{x}(i,j) \), one can instead consider the stability of an equivalent filter realization (\( \bar{A}_1, \bar{A}_2, \bar{b}_1, \bar{b}_2, \bar{c}, \bar{d} \)) with \( \bar{A}_1 = T^{-1} A_1 T, \quad \bar{A}_2 = T^{-1} A_2 T \). The generalized 2-D Lyapunov equation for state-space representation (\( \bar{A}, \bar{b}, \bar{c}, \bar{d} \)) is

\[
Q = \begin{bmatrix}
P^{T/2} W_1 P^{1/2} & 0 \\
0 & P^{T/2} W_2 P^{1/2}
\end{bmatrix} - \bar{A}^T \bar{P}^{1/2} \bar{R} \bar{P}^{1/2} \bar{A}
\]

and condition (9) becomes

\[
\bar{R} - \bar{W}_1 - \bar{W}_2 \succeq 0
\]

Now let \( T = P^{-1/2} \bar{R}^{1/2} P^{1/2} \) for some \( P > 0 \) and note that

\[
\bar{A} = [\bar{A}_1 \quad \bar{A}_2] = T^{-1} A \begin{bmatrix} T \\ 0 \end{bmatrix}
\]

(20) becomes (18) with

\[
Q = \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix} \hat{Q} \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix}
\]

and

\[
W_1 = \bar{R}^{-T/2} \bar{W}_1 \bar{R}^{-1/2}, \quad W_2 = \bar{R}^{-T/2} \bar{W}_2 \bar{R}^{-1/2}
\]

From (22) and (23) it follows that \( Q > 0 \) if and only if \( \hat{Q} > 0 \), and that (21) is satisfied if (19) holds.

**IV. NUMERICAL SOLUTION TO GENERALIZED LYAPUNOV EQUATION**

In this section we consider the problem of numerically solving the Lyapunov equation (18) in the following sense: Given \( \bar{A} = [\bar{A}_1 \quad \bar{A}_2] \), find \( P > 0 \), \( W_1 > 0 \), and \( W_2 > 0 \) such that \( Q \) defined by (18) is positive definite and (19) holds. In what follows, we first present a result that relates the existence of such positive definite \( P, W_1, W_2 \) to a norm minimization problem. Discussions then follow to consider numerical solution of this norm minimization problem for two special cases where both matrices \( W_1, W_2 \) are diagonal.

**Theorem 4:** There exist \( P > 0 \), \( W_1 > 0 \), and \( W_2 > 0 \) such that (19) holds and \( Q \) defined in (18) is positive definite if and only if

\[
\begin{aligned}
\text{minimize} & \quad \| \hat{Q} \|_{F, V_1, V_2} \\
\text{subject to} & \quad \| \hat{Q} \|_{F, V_1, V_2} \leq 1
\end{aligned}
\]

where \( \hat{A} = [\bar{A}_1 \quad \bar{A}_2], \quad \hat{A}_1 = T^{-1} A_1 T, \quad \hat{A}_2 = T^{-1} A_2 T \) and \( \| \cdot \| \) denotes the induced 2-norm of the matrix involved.

**Proof:** Assume there exist \( P > 0 \), \( W_1 > 0 \), and \( W_2 > 0 \) such that \( Q \) in (18) is positive definite and (19) holds; We write

\[
W_1 = V_1^{-T} V_1^{-1} \text{ and } W_2 = V_2^{-T} V_2^{-1}
\]

and set \( T^{-1} = P^{1/2} \) in order to write (18) and (19) as shown in (25) and

\[
I - V_1^{-T_1} V_1^{-1} - V_2^{-T_2} V_2^{-1} \succeq 0
\]

respectively. As \( Q > 0 \), (25) implies that

\[
\| \hat{Q} \|_{V_1, V_2} = \left[ \begin{array}{c}
V_1 \\
0
\end{array} \right] \begin{bmatrix}
0 & 0 \\
V_2 & 0
\end{bmatrix} = 0
\]

and hence (24) holds. Conversely, if (24) holds then there are some nonsingular \( T \), \( V_1, V_2 \), satisfying (27). This means

\[
I - V_1^{-T_1} V_1^{-1} - V_2^{-T_2} V_2^{-1} \succeq 0
\]

and

\[
\hat{Q} \equiv I_{2n} - \begin{bmatrix} V_1 & 0 \\
0 & V_2
\end{bmatrix} A^T \begin{bmatrix} V_1 & 0 \\
0 & V_2
\end{bmatrix} > 0
\]

If we let \( P = T^{-T_1} T^{-1}, \quad W_1 = V_1^{-T_1} V_1^{-1}, \quad W_2 = V_2^{-T_2} V_2^{-1} \), then (29) implies that \( Q \) defined in (18) is positive definite where

\[
Q = \begin{bmatrix} T^{-T_1} V_1^{-T} & 0 \\ 0 & T^{-T_2} V_2^{-T} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = 0
\]

and (28) gives (19).

There is a variety of efficient methods that can be used to solve the optimization problem (24) [10], [11]. In what follows we discuss two special cases where \( V_1 \) and \( V_2 \) assume diagonal forms, leading to much simplified unconstrained minimization problems. Case A: In this case \( V_1 \) and \( V_2 \) assume the form of

\[
V_1 = \begin{bmatrix}
1/v_{11} & 0 \\
\vdots & \ddots \\
0 & 1/v_{1n}
\end{bmatrix}, \quad V_2 = \begin{bmatrix}
1/v_{21} & 0 \\
\vdots & \ddots \\
0 & 1/v_{2n}
\end{bmatrix}
\]

with \( v_{1i} > 0 \) and \( v_{2i} > 0 \) (1 ≤ i ≤ n). To meet constraint (28), it is required that

\[
v_{1i} + v_{2i} = 1, \quad 1 \leq i \leq n
\]

From (22) and (23) it follows that \( Q > 0 \) if and only if \( \hat{Q} > 0 \), and that (21) is satisfied if (19) holds.
which implies that
\[ 0 < v_i < 1 \]
\[ v_{2i} = (1 - v_i)^{1/2} \]

The hyperbolic tangent transformation
\[ v_i = \frac{1 + \tanh \nu_i}{2}, 1 \leq i \leq n \]
can be used to meet constraint (32) where \( \nu_i \in (-\infty, \infty) \).

With (30), (33) and (34), the minimization problem is simplified as
\[ \text{minimize} \left\| A \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \right\| \]

where
\[
V_1 = \begin{bmatrix}
2 & 0 \\
0 & 2 \\
\vdots & \vdots \\
0 & 0 \\
\end{bmatrix}
\]
\[
V_2 = \begin{bmatrix}
[4 - (1 + \tanh \nu_1)^2]^{1/2} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & [4 - (1 + \tanh \nu_n)^2]^{1/2} \\
\end{bmatrix}
\]

(36)

Although the state-variable transformation matrix \( T \) in problem (35) needs to be nonsingular, (35) can be considered as an unconstrained optimization problem due to the following two reasons: First, the measure of the set of points in the parameter space that make \( T \) singular is zero from a classic measure-theoretic point of view. Second, during the minimization, updating a parameter vector to a new one that would be closer to a singular \( T \) is always discouraged as a nearly singular \( T \) tends to increase the value of the norm in (35) due to the presence of \( T^{-1} \) in \( A \). In other words, the minimization problem is the use of a good optimization algorithm would very likely lead to a solution that corresponds to a nearly singular \( T \).

**Case B:** In this case \( V_1 \) and \( V_2 \) assume the form of
\[
V_1 = \frac{1}{v_i} I \quad \text{and} \quad V_2 = \frac{1}{v_2} I
\]

with \( v_i > 0 \) \((i = 1, 2)\). To satisfy (28), it is required that
\[ v_1^2 + v_2^2 = 1 \]

Hence
\[ 0 < v_1 < 1 \quad \text{and} \quad v_2 = (1 - v_1^2)^{1/2} \]

Similar to Case A, an hyperbolic tangent transformation
\[ v_1 = \frac{1 + \tanh \nu}{2}, -\infty < \nu < \infty \]

and
\[ v_2 = \frac{1}{2}[4 - (1 + \tanh \nu)^2]^{1/2}, -\infty < \nu < \infty \]

can be used in (37) with \(-\infty < \nu < \infty\) so that the minimization problem
\[ \text{minimize} \left\| A \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \right\| \]

with \( V_i \) \((i = 1, 2)\) specified by (37)-(39) is essentially unconstrained.

Finally, it is easy to see that as \( W_1 = V_1^{-1} V_1^{-1} = v_i I \) and \( W_2 = V_2^{-1} V_1^{-1} = v_2 I \), (40) with (37)-(39) provides a feasible approach to numerically solving Hinamoto’s Lyapunov equation (3).

### V. An Example

In this section we use a second-order state-space digital filter modeled by (1) with
\[
A_1 = \begin{bmatrix} 0.8 & 1 \\ 0 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.8 \end{bmatrix}
\]
to illustrate the results obtained in the preceding sections. Note first that
\[
\det(I - z_1 A_1 - z_2 A_2) = (1 - 0.8z_1 - 0.1z_2)(1 - 0.1z_1 - 0.8z_2)
\]

For \((z_1, z_2) \in \mathbb{C}^2\)
\[
[1 - 0.8z_1 - 0.1z_2] > 1 - 0.8 |z_1| - 0.1 |z_2| > 1 - 0.8 - 0.1 > 0
\]
\[
[1 - 0.8z_1 - 0.1z_2] > 1 - 0.1 |z_1| - 0.8 |z_2| > 1 - 0.1 - 0.8 > 0
\]

thus \(\det(I - z_1 A_1 - z_2 A_2) \neq 0\) for \((z_1, z_2) \in \mathbb{C}^2\) and, therefore, the filter is asymptotically stable [9].

Further notice that
\[
\det(I - z_1 A_1 - z_2 A_2) = 0 \quad \text{at} \quad P_1 : (z_1, z_2) = (1.25, 0)
\]
and
\[
P_2 : (z_1, z_2) = (0.125)
\]

Since points \(P_1\) and \(P_2\) correspond to locations in areas \(S_1\) and \(S_2\) (see Fig. 2), respectively, we know from Sec. II that no \(\alpha, \beta\), and \(P > 0\) can be found to make \(Q\) defined in (3) positive definite with \(\alpha, \beta\) satisfying (4). As a matter of fact, with initial \(T = I\) and \(\nu = 0.4\), solving minimization problem (40) in this case gives
\[ \text{minimize} \left\| A \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \right\| \approx 1.14065 \]

where the minimum is achieved by
\[
T = \begin{bmatrix} 1.68421 & -0.17853 \\ 0.00224 & 0.01781 \end{bmatrix} \quad \text{and} \quad \nu = 0.44077
\]

Although the objective function in (40) has many local minimum points, a large number of trials with randomly chosen initial points indicates that the global minimum value of the objective function is no less than 1.14, which obviously agrees with the conclusion made before.

We now consider the generalized Lyapunov equation (15) with constraint (14). This amounts to solving the minimization problem (35), (36). With initial \(T = I\) and \(\nu_1 = \nu_2 = 0.4\), we obtain
\[ \text{minimize} \left\| A \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \right\| = 0.99057 < 1 \]

where the local minimum is achieved by
\[
T = \begin{bmatrix} 1.78522 & -0.02254 \\ 0 & 0.00098 \end{bmatrix} \quad \text{and} \quad \nu_1 = 1.42871, \quad \nu_2 = -0.38240
\]

It follows from Theorem 4 that
\[
P = T^{-T}T^{-1} = \begin{bmatrix} 0.31381 & 9.540066 \\ 0.540066 & 1.040436 \times 10^3 \end{bmatrix} > 0
\]
\[
W_1 = V_1^{-1} V_1^{-1} = \begin{bmatrix} 0.59435 & 0 \\ 0 & 0.106872 \end{bmatrix} > 0
\]
\[
W_2 = V_2^{-1} V_1^{-1} = \begin{bmatrix} 0.05649 & 0 \\ 0 & 0.899128 \end{bmatrix} > 0
\]

satisfy constraint (14). It is found that \(Q\) defined in (15) is given by
\[
Q = \begin{bmatrix} 0.079814 & 5.671777 \\ 5.671777 & 9.46706 \times 10^3 \end{bmatrix}
\]

which is positive definite. Therefore Corollary 2.2 confirms the stability of the filter.
VI. CONCLUSIONS

We have in this paper made a generalization of the constant 2-D Lyapunov equation proposed recently by Hinamoto [8] and showed its usefulness in stability analysis of 2-D digital filters in a local state-space setting.

REFERENCES


Interaction of Low- and High-Frequency Oscillations in a Nonlinear RLC Circuit

Ali Oksasoglu and Dimitry Yavriv

Abstract—The interaction of low- and high-frequency oscillations in an RLC circuit with a nonlinear capacitance is studied from the point of view of the modern theory of dynamical systems. It is found that for a certain range of parameters such an interaction may cause chaotic instability, even under the weakly nonlinear excitation conditions. The scenarios and conditions that give rise to chaotic oscillations are investigated both numerically and analytically.

I. INTRODUCTION

Today, the nonlinear reactances, such as varactor diodes, have a wide range of use in many areas of electrical engineering. The nonlinear capacitances are incorporated into a circuit so as to design parametric amplifiers, up-converters, mixers, low-power microwave oscillators, electronic tuning devices, etc. A simple RLC circuit with

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a nonlinear reactance, as shown in Fig. 1, can be considered to be a typical building block for some of the above-mentioned devices. The circuit contains two sources; the input source, $v_i(t) = V_m \sin(\omega t)$, is the high-frequency signal, and $E(t) = E_m \sin(\Omega t)$ is the low-frequency voltage source used to modulate the nonlinear capacitor. By the low- and high-frequency distinction, we mean $\omega \gg \Omega$. So far, such circuits have been studied mainly by using the method of harmonic balance or some other similar methods. However, in the frame of such methods, it is not possible to give an exact account of the stability performance of the system. Recent results obtained in the theory of dynamical systems [1] indicate that the dynamics of even such simple circuits should be revisited from the point of chaotic instability. In this paper, we investigate the interaction of low and high frequencies in the circuit, and show that such an interaction can initiate chaotic instability.

II. FORMULATION

For the $v - q$ characteristic of the nonlinear capacitance, let us use the approximation $v_c = q/C_0 + a_2q^2 + a_3q^3$ where $C_0$ is the value of the "linear" capacitance, and $a_2$, and $a_3$ are the nonlinearity coefficients depending on the type of the varactor diode used. In this case, the differential equation governing the circuit’s behavior is reduced to the equation of motion of the Duffing type oscillator:

$$
\frac{d^2q}{dt^2} + \omega_0^2q = \frac{-a_2}{L}q - \frac{a_3}{L^3}q^3 - \frac{R}{L} \frac{dq}{dt} + \frac{V_m}{L} \sin(\omega t)
+ \frac{E_m \sin(\Omega t)}{L}
$$

(1)

where $\omega_0^2 = 1/LC_0$ is the natural frequency of the oscillator. We consider the resonant excitation when $|\omega - \omega_0| \approx \omega_0/Q$, where $Q = \omega_0L/R$ is the quality factor of the circuit.

So far, the chaotic behavior of such oscillators has been repeatedly demonstrated for the case of its strong nonlinearity under large external perturbations, i.e., large $V_m$ (see e.g. [1], [2]). However, such situations are not typical for most practical realizations. In practice, $V_m$ is usually small in the sense that such systems can be considered weakly-nonlinear, and according to the previous results obtained in this area, the chaotic instability is not typical for such systems.

Let us consider the interaction of the low and high frequency oscillations under weak nonlinearity conditions. By using the transformation $q = q_0 + E(x + E_0)^2$, where $q_0$ is the unit charge, and $x$ is a dimensionless variable, (1) can be reduced to the quasi-linear form,

$$
\frac{dx}{dt} + x = -2b(\frac{dz}{dt} + b_0(x + E_0)^2 + b_3(x + E_0)^3 - V_0 \sin(\nu t))
$$

(2)